Performance Evaluation with High Moments and Disaster Risk*

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Abstract

Tail events and rare disasters have been fundamental to the recent financial crisis. However, traditional performance evaluation measures do not account for such events, even when they are present in the data. To address this issue, we propose reinterpreting the recent riskiness measures of Aumann and Serrano (2008) and Foster and Hart (2009) as performance indices. We derive the high moment properties of these indices and show that they are increasing in all odd moments (e.g., mean and skewness) and decreasing in all even moments (e.g., variance and tail-risk). The sensitivity of the performance indices to the moments is not monotone, implying that high moments can have a strong effect on performance. Furthermore, we show that the Foster and Hart (2009) index is extremely sensitive to the risk of rare disasters, making it a useful tool for evaluating such risks. We illustrate the applicability of these indices by using them to evaluate popular anomalies and investment strategies, and by applying them to the selection of mutual funds. Our analysis demonstrates that high moment properties and disaster risk are important factors in performance evaluation.

1 Introduction

Tail risk and rare disasters have been central to the recent meltdown in financial markets. Indeed, markets were hit by catastrophic events whose ex-ante probabilities were considered negligible. Traditional performance evaluation measures (such as the Sharpe ratio) typically rely on the first two distribution moments, thereby underestimating the effects of rare disasters. Indeed, low distribution moments hardly account for rare and catastrophic events, since their large negative effect is multiplied by a very small probability. By contrast, when one considers high distribution moments, an extremely negative but rare outcome is raised to a high power, making its effect on the moment substantial regardless of the small probability associated with it.

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High distribution moments have received notable attention in the asset pricing literature. In particular, a large body of work in asset pricing suggests that investors favor right skewness (e.g., Kraus and Litzenberger (1976), Jean (1971), Kane (1982), and Harvey and Siddique (2000)), but are averse to tail-risk and rare disasters (e.g., Barro (2009) and Gabaix (2008)). It is thus desirable that normative performance evaluation measures reflect these preferences. Yet, to the best of our knowledge, there is no performance evaluation measure that incorporates these high moment and disaster risk properties.

In this paper we study two such performance indices which are based on a simple reinterpretation of the novel riskiness measures proposed recently by Aumann and Serrano (2008) and Foster and Hart (2009) (hereafter AS and FH, respectively). We investigate the moment properties of these indices and establish that they reflect all distribution moments in a manner that is consistent with economic intuition and with the asset pricing literature. We also study the way these two indices reflect disaster risk, and show that they differ in their sensitivity to such risks. We then apply these indices to popular investment strategies and to well-known anomalies, show their practical usefulness in selecting mutual funds, and demonstrate the pitfalls associated with ignoring high moments and rare disasters in performance evaluation.

Our starting point is that investors are risk-averse and choose their investments by maximizing expected utility. Given that the exact form of the utility function of each investor is not observable, a desirable performance evaluation measure should fit all plausible utility functions. The best possible way to rank investments in this setup is known to be Second Order Stochastic Dominance (SOSD) (See Hadar and Russell (1969), Hanoch and Levy (1969), and Rothschild and Stiglitz (1970)), according to which one investment dominates another if all risk averse investors prefer the former to the latter.\(^1\) The problem with SOSD is that it only imposes a partial order on investments. Namely, picking two investments, it is often the case that neither of them dominates the other based on SOSD.

Based on our discussion thus far, a desirable performance evaluation index should satisfy the following four requirements (i) It imposes a complete order on investments, namely, any two investments can be compared; (ii) It depends on the distribution of outcomes only. That is, the form of the utility function or the wealth of particular investors is not needed to calculate the performance index; (iii) It coincides with SOSD, whenever SOSD can be applied. Namely, if all risk averse investors prefer one investment to the other, then the performance index ranks the investments accordingly; and (iv) It accounts for high distribution moments in a manner consistent with the asset pricing literature. That is, the index is increasing in mean and skewness and decreasing in variance and tail-risk of the investment.

The Sharpe ratio, which is probably the most popular performance evaluation measure, satisfies (i) and (ii). It clearly fails (iv) since it only reflects the first two distribution moments. Interestingly, it also fails (iii). Indeed, it is relatively easy to find examples where all risk averse investors prefer one investment to the other and

\(^1\)A popular example of SOSD is when one investment differs from another by a mean preserving spread. See Rothschild and Stiglitz (1970).
yet the Sharpe ratio ranks the investments in the wrong order (see Section 2 for one such example).

To understand the fundamental insights in AS and FH it is useful to follow the approach presented in Hart (2011), who offers a unified framework for the two. The key for the new measures is to use the investor’s initial wealth as a benchmark for her investment decisions. That is, instead of comparing the expected utility of two investments, we compare the expected utility of each investment separately to the status quo, and ask which one of the two investments is uniformly rejected more often. If every time that investment $g$ is uniformly rejected we have that investment $g'$ is also uniformly rejected, then $g$ is deemed more attractive than $g'$ (i.e., $g$ has better performance than $g'$).\(^2\) That is, $g$ is more attractive than $g'$ if $g$ is rejected “less often” than $g'$ in some uniform manner when compared to the status quo.

The term “uniformly rejected” can take two different meanings. First is “wealth-uniform rejection” in which for a given utility function, an investor rejects the investment relative to the status quo for all wealth levels. Second is “utility-uniform rejection” in which for a given wealth level, all utility functions reject the investment relative to the status quo. The former approach to uniform rejection leads to the AS performance index, while the latter leads to the FH performance measure.

As shown in AS, FH, and Hart (2011), the two approaches yield two rankings of all possible investments, each of which can be represented by a positive performance index that possesses a simple and intuitive economic interpretation. Both indices satisfy requirements (i)-(iii) above. Moreover, they can be easily calculated from the distribution of the investment by solving a simple and intuitive implicit equation. The only difference between our interpretation and the interpretations given in AS and FH is that they choose to consider the riskiness of the investment, deeming one investment “more risky” than another if it is uniformly rejected more often relative to the status quo. We choose to focus on the flip side of the argument, viewing one investment as “more attractive” if risk-averse investors show little aversion to this investment when compared to the status quo, in a uniform manner.

The first thing we do in this paper is to study how the AS and FH performance indices are affected by the moments of the investments being evaluated. In contrast to traditional performance measures, we establish that both the AS and FH indices reflect all the distribution moments. Moreover, these performance indices are increasing in all odd moments and decreasing in all even moments.\(^3\) Consequently, the two performance evaluation indices satisfy requirement (iv) above. Intuitively, the performance evaluation indices can be viewed as a function that encapsulates all distribution moments. This function is increasing in all odd moments and decreasing

\(^2\)The term “investment” here simply refers to a random variable which can be described by the probability distribution of outcomes. Occasionally we will use the term “gamble”, which is the one used in AS and FH, instead. We use the letter $g$ as a generic notation for such investments (or gambles).

\(^3\)We use the term “moments” as a shorthand for both raw and central moments. All of the properties we discuss apply to both types of moments.
Next, we ask whether the sensitivity of the performance indices to the moments is monotonically decreasing in the moment rank. Namely, is it the case that high distribution moments necessarily have a smaller effect on performance than low distribution moments, and thus should be neglected (as is often done in practice). We establish that there is no such monotone relation. In particular, the performance indices can be either more or less sensitive to higher moments. Thus, high moments can have a material effect on performance, and should not be neglected.

We then turn to exploring how the performance measures are affected by rare disasters, modeled as extremely negative outcomes associated with vanishing probabilities. First, note that such outcomes tend to make the distribution left skewed (lower third moment) and fat-tailed (higher fourth moment). Thus, given requirement (iv), the two performance measures are adversely affected by rare disasters. However, we show that there is a substantial difference in the sensitivity of the two measures to rare disasters, where the FH index is much more sensitive than the AS index. To formalize this point we consider an outcome distribution that can be divided into two parts: (i) a “business as usual” part, that accounts for the vast majority of the distribution; and (ii) a “rare disaster,” which is a very negative outcome $-L$, associated with a very small probability. We show that the FH measure in this case is approximately equal to $1/L$. That is, in the presence of a rare disaster, the catastrophic outcome “consumes” the FH performance index, which becomes roughly equal to 1 over the absolute value of the catastrophic return. This property is not shared by the AS index. While the latter is sensitive to rare disaster by the virtue of requirement (iv), it is not “consumed” by those events. Thus, in practice one should use one index or the other depending on the desirable sensitivity of the index to rare disasters.

We next turn to exploring the practical implications of the two performance indices. To this end, we show that the two indices lend themselves naturally to estimation using the Generalized Method of Moments (GMM) (see Hansen (1982)). The reason is that the implicit equations used to solve for the indices are naturally interpreted as moment conditions for the GMM estimation. GMM not only facilitate the estimation of the indices, but it also works out a distribution for the index estimates which is consistent, efficient, and asymptotically normal. Thus, we can use these estimates to test hypotheses regarding the attractiveness of different investment strategies in the underlying population of returns.

We first use the performance indices to evaluate the most prominent and widely studied investment anomalies in the finance literature: the size anomaly, the value anomaly, and the momentum anomaly. We compare these investment strategies to each other and to a naive “buy and hold” strategy of investing in the market portfolio. We do this by examining the performance of the four Fama-French factors (Fama and French (1993)), which are essentially portfolios constructed based on the market portfolio and the three anomalies. Our most interesting finding here is that the momentum strategy, which is often considered the most serious and hard to explain deviation from market efficiency (Fama and French (1996)), is not an attractive
investment strategy when accounting for high moments. Momentum returns are extremely left skewed and fat tailed, and they exhibit extreme negative events, which fall under our definition of “rare disasters.” These high moment properties outweigh the higher average return obtained from following momentum. In particular, our GMM estimates of the AS and FH performance estimates allow us to reject the hypothesis that momentum has better performance than a “buy and hold” investment in the market portfolio. Moreover, when using the FH index (which is very sensitive to rare disasters), we find that momentum is dominated by the value anomaly.

In our next application we compare the performance of private investments to public equity. Moskowitz and Vissing-Jorgensen (2002) find that the returns to private equity are not higher than those of public equity. They view this result as puzzling since private equity investments expose investors to a high level of idiosyncratic risk. Moskowitz and Vissing-Jorgensen note that private equity returns are right skewed and conjecture that preference for skewness may be one reason for the tendency of individuals to invest in private equity. The performance measures studied in this paper are useful for evaluating this statement since they take into account all distribution moments (skewness among them). Thus, we follow Moskowitz and Vissing-Jorgensen (2002) and compare the returns of public investments to those of private investments obtained from the 2004 Survey of Consumer Finance (SCF). We find that the average return on private equity conditional on survival is about 35 times larger than that of public equity. Moreover, private equity returns are indeed very right skewed. However, private equity returns are also extremely more volatile and fat-fat tailed than the returns on public equity. The question is then whether the superior first and third moments of private equity outweigh its inferior second and fourth moments. The GMM estimates of the two performance measures suggest that this is not the case. Both the AS and FH measures are significantly higher for public investments. Thus, based on our estimates, the “private equity premium puzzle” suggested by Moskowitz and Vissing-Jorgensen (2002) still stands, and is not resolved by high moment properties.

In the next application we compare the performance of actively managed equity funds to the performance of index funds. The debate on the value of active as opposed to passive management of mutual funds dates back to Jensen (1968). One aspect of this debate that has not received much attention is the high moment properties of the portfolios selected by active mutual funds when compared to index funds. We examine this issue and find that the moments of the two management strategies are not materially different. Moreover, our estimates show that the performance indices of active vs. passive mutual funds (after accounting for fees) are not significantly different. Thus, the new performance indices reinforce the view that active management does not improve investment performance (e.g., Malkiel (1995)).

In our final exercise we take the two performance indices one step farther. Rather than just examining the performance of investment strategies, we use the indices to select among actively managed mutual funds, and examine the performance resulting from such an investment strategy. In each month during our sample period of 1967-2009 we rank all actively managed equity mutual funds based on their AS and FH
measures (calculated over a rolling 60 month window). We then obtain two portfolios of \textquotedblleft selected\textquotedblright mutual funds by equal-weighting the funds in the top decile for each index. We rebalance this portfolio on a monthly basis.

We compare these active portfolios selected based on the two indices to the market portfolio and to a portfolio selected based on the Sharpe ratio. We find that moments generated by the AS and FH measures are significantly more appealing than those generated by the Sharpe ratio and are also often more appealing than those of the market portfolio. In particular, portfolios of mutual funds based on the AS and FH indices have lower variance, less negative skewness, and lower tail risk than the market or the Sharpe ratio based portfolios. The two performance indices estimates reflect these observations. Indeed, both the AS and FH indices are higher for portfolios which select mutual funds based on these two indices. This suggests that the two indices may be useful not only for evaluating investments but also for selecting investments that have desirable moment properties.

The rest of the paper is organized as follows. Section 2 presents motivating examples. Section 3 introduces the two performance measures by reviewing relevant results from AS, FH, and Hart (2011). In Section 4 we derive the high moment and rare disaster properties of the two performance indices. Section 5 discusses practical applications of the performance indices to different investment strategies. Section 6 studies the behavior of mutual fund portfolios selected using the two indices. We conclude in Section 7.

2 Motivating Examples

Before introducing the new performance measures we first examine a few examples of performance evaluation based on the traditional Sharpe ratio. These examples highlight important distributional features that fail to be captured by the calculation of the Sharpe ratio. In contrast, we will show later in the paper that the new performance indices successfully incorporate these features.

The first example, as shown in Table 1, involves the comparison of two gambles. Gamble $g_1$ looks like a relatively safe bet. However, it assigns a very small probability to a rare but disastrous event of losing 10. By contrast, $g_2$ is more volatile than $g_1$. Yet, the distribution of $g_2$ lies (weakly) to the right of that of $g_1$. Hence, $g_2$ first-order stochastically dominates $g_1$. That is, all investors with increasing utility prefer $g_2$ to $g_1$. Yet, the Sharpe ratio of $g_1$ is higher than that of $g_2$. This reflects the fact that the variance of $g_1$ is very low. In this case, the Sharpe ratio fails to capture the preference of any risk-averse (or even risk-loving) investor.

The problem with the Sharpe ratio is tied closely to the high moments of the gambles. Indeed, both the mean and the variance of $g_1$ are only mildly affected by the rare disaster. However, higher moments would be more strongly affected by this event. Indeed, by the way higher moments are calculated, a disastrous outcome is raised to a higher and higher power, while the probability associated with it does not change. As a result, for sufficiently high moments, disastrous outcomes will dominate the low probability assigned to them, and hence will have a material effect on the
moment itself. Investors maximizing expected utility care about all moments of the distribution (not just the first two). Hence, disastrous but rare events such as in $g_1$ may have a material effect on their preferences.

In the example above we have calculated the third and fourth central moments of $g_1$ and $g_2$ for illustration (denoted by $m_3$ and $m_4$). It can be seen that the third moment of $g_1$ is negative while the third moment of $g_2$ is positive, reflecting the left skewness of $g_1$ compared to $g_2$. And, the fourth moment of $g_1$ is larger than that of $g_2$, reflecting the tail risk associated with $g_1$. These high moments are incorporated into the decisions of expected utility maximizers. Our view is that they should also be incorporated into performance evaluation indices.

The second example is summarized in Table 2. With a large probability gamble $g_1$ yields an outcome superior to all possible realizations of $g_2$. However, it also assigns a very small probability to a disastrous event of losing 8. The rare disaster does not strongly affect the first moment ($\mu_1$), but it starts to exert a larger and larger effect on higher moments, as reflected by a larger variance, left skewness and higher tail risk of $g_1$. Unlike in the previous example, we no longer have first-order stochastic dominance in this case. We are interested in seeing whether the Sharpe ratio is able to capture the rare disaster and the associated high moment properties when first-order stochastic dominance does not work.

This example shows that gamble $g_1$ has a higher Sharpe ratio than $g_2$, with the larger variance dominated by the higher mean return. That is, the Sharpe ratio suggests $g_1$ to be a better gamble than $g_2$. Apparently, the Sharpe ratio fails to capture the rare disaster and the associated high moment properties even when first-order stochastic dominance does not work.

### 3 The Performance Indices

An investment can be modeled as a random variable, which we generically denote by $g$. We assume that all of the moments of $g$ are well defined. Furthermore, we assume that $g$ has positive expectation and that it admits some negative values with a positive probability. We follow Hart (2011) in assuming that $g$ takes finitely many values. We often refer to $g$ as a “gamble” and denote the set of all possible gambles by $\mathcal{G}$.

We assume that investors have Von Neumann-Morgenstern utility functions over wealth denoted by $u(\cdot)$, which are differentiable as many times as needed. We assume further that $u' > 0$ and $u'' < 0$, reflecting that investors like more wealth over less, and are strictly risk-averse. Furthermore, we restrict attention to utility functions $u$ satisfying the following three conditions: (i) Decreasing absolute risk aversion, i.e. $-\frac{u''(w)}{u'(w)}$ is weakly decreasing; (ii) Increasing relative risk aversion, i.e. $-w u''(w) u'(w)$ is weakly increasing; and (iii) $\lim_{w \to 0} u(w) = -\infty$. We denote the class of all such utility functions by $\mathcal{U}^*$, and note that this class includes, for example, all CRRA utility functions of the form $u(w) = \frac{w^{1+\gamma}}{1+\gamma}$ with $\gamma \geq 1$, as well as CARA utility functions with sufficiently high wealth.
Let \( w_0 \) denote the initial wealth of an investor, to which we refer as her “status-quo.”

**Definition 1** Say that an investor with utility \( u \) and initial wealth \( w_0 \) rejects a gamble \( g \) if \( E u (w_0 + g) \leq u (w_0) \), and accepts a gamble \( g \) if \( E u (w_0 + g) > u (w_0) \).

That is, an investor rejects a gamble whenever her status-quo yields her a higher expected utility. The following two definitions are needed to describe the Aumann-Serrano performance index.

**Definition 2** Say that a gamble \( g \) is wealth-uniformly rejected by an investor with utility function \( u \) if \( u \) rejects \( g \) at all initial wealth levels \( w_0 \).

Intuitively, an investor wealth-uniformly rejects a gamble \( g \), if she prefers the status-quo to \( g \), regardless of her wealth level.

**Definition 3** Say that a gamble \( g \) wealth-uniformly dominates gamble \( g' \) if whenever \( g \) is wealth-uniformly rejected by a utility function \( u \), \( g' \) is also wealth-uniformly rejected by \( u \).

Namely, \( g \) wealth-uniformly dominates \( g' \) if whenever an investor with utility function \( u \) prefers the status-quo to \( g \) for all wealth levels, she also prefers the status-quo to \( g' \) for all wealth levels. In other words, \( g \) is preferred to \( g' \), if \( g' \) is “more often” wealth-uniformly rejected than \( g \) is.

**Theorem 1** (Aumann and Serrano (2008), Hart (2011)). Wealth-uniform dominance induces a complete order on \( \mathcal{G} \) that extends SOSD. This order can be represented by a performance index \( P_{AS} (g) \) assigned to any gamble \( g \), which is given by the unique positive solution to the implicit equation

\[
E \left[ \exp \left( -P_{AS} (g) g \right) \right] = 1. \tag{1}
\]

That is, for any two gambles \( g \) and \( g' \), \( g \) wealth-uniformly dominates \( g' \) if and only if \( P_{AS} (g) \geq P_{AS} (g') \).

To gain intuition for the performance measure \( P_{AS} \) it is useful to rewrite (1) as

\[
E \left[ -\exp \left( -P_{AS} (g) (w_0 + g) \right) \right] = -\exp \left( -P_{AS} (g) w_0 \right), \tag{2}
\]

for some initial wealth \( w_0 \). Note that (1) and (2) are equivalent regardless of \( w_0 \). Thus, a useful interpretation is that \( P_{AS} (g) \) is the level of absolute risk aversion that makes an investor with CARA utility indifferent between taking \( g \) and the status quo, regardless of the initial wealth \( w_0 \). Put differently, an investor with CARA utility \( u (w) = -\exp (-\lambda w) \), would accept \( g \) when \( \lambda < P_{AS} (g) \), and would reject \( g \) when \( \lambda \geq P_{AS} (g) \). Thus, a higher level of \( P_{AS} (g) \) means that investors are “less averse” to \( g \), in the sense that a higher level of risk aversion is needed to reject \( g \). The key insight in Theorem 1 is that checking (1) is both necessary and sufficient for
wealth-uniform dominance for all utility functions in $U^*$. As such, a higher level of $P^{\text{AS}}(g)$ corresponds to better performance for all utility functions in $U^*$.

The next two definitions are needed to describe the Foster-Hart performance index.

**Definition 4** Say that a gamble $g$ is utility-uniformly rejected at an initial wealth level $w_0$ if all utility functions $u \in U^*$ reject $g$ at $w_0$.

That is, a gamble $g$ is utility-uniformly rejected at wealth level $w_0$, if any investor, regardless of her utility function, prefers the status-quo to $g$ at $w_0$.

**Definition 5** Say that a gamble $g$ utility-uniformly dominates gamble $g'$ if whenever $g$ is utility-uniformly rejected at an initial wealth level $w_0$, $g'$ is also utility-uniformly rejected at $w_0$.

Namely, $g$ utility-uniformly dominates $g'$ if whenever all investors with initial wealth level $w_0$ prefer the status-quo to $g$, they also prefer the status-quo to $g'$. Roughly, $g$ is preferred to $g'$, if $g'$ is “more often” utility-uniformly rejected than $g$ is.

**Theorem 2** (Foster and Hart (2009), Hart (2011)). Utility-uniform dominance induces a complete order on $G$ that extends SOSD. This order can be represented by a performance index $P^{FH}(g)$ assigned to any gamble $g$, which is given by the unique positive solution to the implicit equation

$$E \left[ \log \left( 1 + P^{FH}(g)g \right) \right] = 0.$$  \hspace{1cm} (3)

That is, for any two gamble $g$ and $g'$, $g$ utility-uniformly dominates $g'$ if and only if $P^{FH}(g) \geq P^{FH}(g')$.

To gain intuition for the performance measure $P^{FH}$ it is useful to rewrite (3) as

$$E \left[ \log \left( \frac{1}{P^{FH}(g)} + g \right) \right] = \log \left( \frac{1}{P^{FH}(g)} \right).$$  \hspace{1cm} (4)

That is, $\frac{1}{P^{FH}(g)}$ can be interpreted as the level of wealth that would render an investor with log utility indifferent between taking $g$ or staying with the status quo. A log investor with higher initial wealth than $\frac{1}{P^{FH}(g)}$ would accept $g$, whereas a log investor with lower initial wealth than $\frac{1}{P^{FH}(g)}$ would reject $g$. Thus, higher $P^{FH}$ corresponds to better performance in the sense that $g$ is accepted even by individuals with lower initial wealth. The key insight in Theorem 2 is that checking (3) is both necessary and sufficient for utility-uniform dominance for all initial wealth levels.

It is worth noting that both Aumann and Serrano (2008) and Foster and Hart (2009) present their measures as “riskiness indices” rather than “performance indices.” However, this is just a matter of interpretation. Their focus is on whether investors are more reluctant to accept one gamble over another, whereas we adopt the traditional performance measurement approach where gambles that investors are
more willing to accept receive a higher score (compare for example to the Sharpe ratio). Given this, the mapping to the original papers (Aumann and Serrano (2008), Foster and Hart (2009), and Hart (2011)) is \( P^{AS} = 1/R^{AS} \) and \( P^{FH} = 1/R^{FH} \), where \( R^{AS} \) and \( R^{FH} \) are the relevant riskiness measures.

Based on the discussion thus far we conclude that the two performance indices \( P^{AS} \) and \( P^{FH} \) satisfy requirements (i)-(iii) in the Introduction. In the next section we study the moment properties of the indices and the way they reflect disaster risk. In particular, we establish that they also satisfy requirement (iv).

4 Moment Properties of the Performance Indices

4.1 Basic Moment Properties

For any gamble \( g \in \mathcal{G} \), let \( \mu_n (g) = \mathbb{E} (g^n) \) be the \( n^{th} \) raw moment of \( g \) \((n \geq 1 \text{ an integer})\) and let \( m_n (g) = \mathbb{E} [(g - \mu_1 (g))^n] \) be the \( n^{th} \) central moment of \( g \) \((n \geq 2 \text{ an integer})\). Since \( g \in \mathcal{G} \), all these moments exist.

In general, any two gambles may differ in several of their moments. However, to get a basic understanding of how different moments are related to the performance indices it is useful to consider the hypothetical exercise of changing one moment at a time while keeping all other moments unchanged. For example, one can think of two investment opportunities that have the same expected return, the same variance of return. Yet, one investment opportunity is more skewed than the other (and all other moments are the same). How does this affect the performance indices of the two investments?

To see how the moments of a gamble affect its performance indices we consider first the \( P^{AS} \) measure. Start by rewriting Eq. (1) as a Taylor expansion around zero:

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left( P^{AS} (g) \right)^n \mu_n (g) = 0.
\]

(5)

Thus, \( P^{AS} (g) \) is given implicitly by the sum of a power series with coefficients proportional to the raw moments of the distribution of \( g \). Odd moments are assigned negative weights, while even moments are assigned positive weights. A similar relation can be written with the central moments using a Taylor series around \( \mu_1 (g) \):

\[
1 + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \left( P^{AS} (g) \right)^n m_n (g) = \exp \left( P^{AS} (g) \mu_1 (g) \right).
\]

(6)

In the next theorem we use these representations to show that the \( P^{AS} \) measure is increasing in all odd moments (both raw and central) and decreasing in all even moments (both raw and central).

**Theorem 3** Consider two gambles \( g, g' \in \mathcal{G} \) and let \( k \) be some positive integer.

1. Assume that for all \( n \neq k \), \( \mu_n (g) = \mu_n (g') \) but \( \mu_k (g) > \mu_k (g') \). Then, \( P^{AS} (g) > P^{AS} (g') \) if \( k \) is odd, while \( P^{AS} (g) < P^{AS} (g') \) if \( k \) is even.
2. Assume that \( \mu_1(g) = \mu_1(g') =: m \) and that for all \( n \neq k \), \( m_n(g) = m_n(g') \) but \( m_k(g) > m_k(g') \). Then, \( P^{AS}(g) > P^{AS}(g') \) if \( k \) is odd, while \( P^{AS}(g) < P^{AS}(g') \) if \( k \) is even.

Next we consider the \( P^{FH} \) measure following the same approach as above. Start by rewriting Eq. (3) as a Taylor expansion around zero:

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (P^{FH}(g))^{n} \mu_n(g) = 0. \tag{7}
\]

Notice that this Taylor expansion converges only when \( -1 < P^{FH}(g) \leq 1 \). As before, the \( P^{FH} \) index is also given implicitly by the sum of a power series with coefficients proportional to the raw moments of the distribution of \( g \). However, odd moments are now assigned positive weights, whereas even moments are assigned negative weights. A similar relation can be written with the central moments using a Taylor series around \( \mu_1(g) \):

\[
\log (1 + P^{FH}(g) \mu_1(g)) = \sum_{n=2}^{\infty} \frac{(-1)^{n}}{n} \left( \frac{P^{FH}(g)}{1 + P^{FH}(g) \mu_1(g)} \right)^{n} m_n(g). \tag{8}
\]

This expansion converges for all \( g \in G \) such that \( -1 < P^{FH}(g) \leq 1 + 2P^{FH}(g) \mu_1(g) \).

The next theorem shows that the \( P^{FH} \) measure is also increasing in all odd moments (both raw and central) and decreasing in all even moments (both raw and central).

**Theorem 4** Consider two gambles \( g, g' \in G \) and let \( k \) be some positive integer.

1. Assume that for all \( n \neq k \), \( m_n(g) = m_n(g') \) but \( \mu_k(g) > \mu_k(g') \). Suppose that \( -1 < P^{FH}(g) \leq 1 \). Then, \( P^{FH}(g) > P^{FH}(g') \) if \( k \) is odd, while \( P^{FH}(g) < P^{FH}(g') \) if \( k \) is even.

2. Assume that \( \mu_1(g) = \mu_1(g') \) and that for all \( n \neq k \), \( m_n(g) = m_n(g') \) but \( m_k(g) > m_k(g') \). Suppose that \( -1 < P^{FH}(g) \leq 1 + 2P^{FH}(g) \mu_1(g) \). Then, \( P^{FH}(g) > P^{FH}(g') \) if \( k \) is odd, while \( P^{FH}(g) < P^{FH}(g') \) if \( k \) is even.

Theorems 3 and 4 tell us among other things that \( P^{AS} \) and \( P^{FH} \) are increasing in the mean and decreasing in the variance of a gamble, which is consistent with traditional performance measures, in particular the Sharpe ratio. In fact, it is shown in Aumann and Serrano (2008) that when a gamble \( g \) has a normal distribution, \( P^{AS}(g) = 2\mu_1(g)/m_2(g) \). That is, in the normal case the \( P^{AS} \) index is proportional to a mean-to-variance ratio. For general distributions, both indices admit larger values when the third moment is large and when the fourth moment is small. Thus, requirement (iv) is satisfied by both indices.
4.2 Magnitude of the Moment Properties

Having established the basic moment properties, we now turn to studying the magnitude of their effects. It is tempting to think that higher moments are less important than lower moments. Indeed, standard performance indices such as the Sharpe ratio ignore completely all moments above the second moment, implicitly assuming that they should be assigned a negligible weight in the performance measure. It is interesting to examine whether the weight assigned to the different moments in the new performance indices is monotonically decreasing in the order of the moment.

A difficulty in examining the relative importance of the moments is that each moment is stated in a different unit of measurement. For example, suppose that the first moment (the mean) is measured in percentage points, then the second moment is measured in percentage points squared, etc. To account for this fact and allow for a “fair” comparison we examine the magnitude effects of the “standardized moments” \( \hat{\mu}_k \equiv \sqrt[k]{\mu_k} \) for \( k = 1, 2, ... \) and \( \hat{m}_k \equiv \sqrt[k]{m_k} \) for \( k = 2, 3, ... \) since all of these have the same units of measurement as the gamble itself. For example, while \( m_2 \) is the variance of the gamble, \( \hat{m}_2 \) is the standard deviation.

To gauge the influence of a moment on the performance index we calculate the elasticity of the index with respect to standardized moments. This gives us a “unit free” estimate of the sensitivity. Since we are focusing on the magnitude of the effects rather than their directions (which we have established already), we will only consider the absolute values of these elasticities (which we term “absolute elasticities”). We begin with studying the absolute elasticity of \( P^{AS} \) and \( P^{FH} \) with respect to \( \hat{\mu}_k \) which we denote by \( \eta^{AS}_k \) and \( \eta^{FH}_k \) respectively.

Consider the \( P^{AS} \) measure first. For all \( g \in \mathcal{G} \), implicitly differentiating (5) yields

\[
\eta^{AS}_k (g) = \frac{1}{(k-1)!} \left( \frac{\hat{\mu}_k}{P^{AS}(g)} \right)^k \frac{|\mu_k (g)|}{\sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!} \left( P^{AS}(g) \right)^n \mu_n (g)}. \tag{9}
\]

Note that \( \eta^{AS}_k \) is homogeneous of degree 0.\(^4\) Hence,

\[
\eta^{AS}_k (g) = \eta^{AS}_k (P^{AS} g) = \frac{1}{(k-1)!} \left[ \mu_k \left( P^{AS} g \right) \right] \frac{1}{\sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!} \mu_n \left( P^{AS} g \right)}.
\]

This normalization allows us to compare the effect of different moments on the performance measure by taking the ratios of the absolute elasticities for different levels of \( k \). Specifically,

\(^4\)This follows because \( \mu_k \) is homogeneous of degree \( k \) and \( P^{AS} \) is homogeneous of degree -1.
\[
\frac{\eta^{AS}_{k+1}(g)}{\eta^{AS}_k(g)} = \frac{1}{(k-1)!} \frac{\mu_{k+1}(P^{AS}g)}{|\mu_k(P^{AS}g)|} \\
= \frac{1}{k} |\mu_{k+1}(P^{AS}g)|^{\frac{1}{k+1}} \left( \frac{|\mu_{k+1}(P^{AS}g)|^{\frac{1}{k+1}}}{|\mu_k(P^{AS}g)|^\frac{1}{k}} \right)^k \\
= \frac{1}{k} |\tilde{\mu}_{k+1}(P^{AS}g)| \left( \frac{|\tilde{\mu}_{k+1}(P^{AS}g)|}{|\tilde{\mu}_k(P^{AS}g)|^k} \right)^k.
\]

If this elasticity ratio takes a value greater than 1, then the \((k+1)\)th moment has a greater effect on the performance measure than the \(k\)th moment. On the other hand, a ratio less than 1 implies that the \(P^{AS}\) measure is less sensitive to the \((k+1)\)th moment as compared to the \(k\)th moment.

To understand the forces that drive this elasticity ratio to be higher or lower than 1, note first that it consists of three components: the factor \(\frac{1}{k}\), \(\tilde{\mu}_{k+1}(P^{AS}g)\), and the ratio of \(\tilde{\mu}_{k+1}(P^{AS}g)\) to \(\tilde{\mu}_k(P^{AS}g)\) raised to the \(k\)th power. First, for all \(k \geq 1\), we have that \(\frac{1}{k} \leq 1\), which drives down the elasticity ratio, and thus the importance of higher moments.

Second, \(|\tilde{\mu}_{k+1}(P^{AS}g)|\) can be higher or lower than 1. If it happens to be the case that \(|P^{AS}g| < 1\) for all realizations of \(g\), then trivially \(\tilde{\mu}_{k+1}(P^{AS}g) < 1\) for all \(k\). Hence, in this case, this term as well drives the importance of higher moments down. Interestingly, in most examples we encounter, \(|P^{AS}g| < 1\) holds (see Sections 5).

Third and perhaps most interesting is that the third factor introduces a force making higher moments more important. To see this note that by Hölder’s inequality,

\[
\tilde{\mu}_{k+1}(P^{AS}|g|) \geq \tilde{\mu}_k(P^{AS}|g|).
\]

This implies that when \(k\) is an odd number,

\[
\left( \frac{\tilde{\mu}_{k+1}(P^{AS}g)}{\tilde{\mu}_k(P^{AS}g)} \right)^k \geq 1.
\]

And, when \(k\) is even,

\[
\left( \frac{\tilde{\mu}_{k+2}(P^{AS}g)}{\tilde{\mu}_k(P^{AS}g)} \right)^k \geq 1.
\]

Thus, the third factor is necessarily greater than 1 for odd values of \(k\). And, for even values of \(k\), we still have a trend up when considering \(k - 2\).

To summarize, somewhat surprisingly, we do not find that higher moments necessarily have a weaker effect on performance evaluation. Rather, we see forces in either direction. In Sections 5 we illustrate this point, showing that higher moments often have a significant effect.
Similarly, for any gamble \( g \in \mathcal{G} \) such that \(-1 < P^{FH}(g) g \leq 1\), we can calculate the absolute elasticity of \( P^{FH} \) with respect to \( \hat{\mu}_k \). The resulting elasticity ratio is given by

\[
\eta_{k+1}^{FH}(g) = \left| \frac{\eta_{k+1}^{FH}(g)}{\eta_{k}^{FH}(g)} \right| ^{k} \left( \frac{\left| \hat{\mu}_{k+1}^{FH}(g) \right|}{\left| \hat{\mu}_{k}^{FH}(g) \right|} \right)^{k}.
\]

Note that compared to the case with the \( P^{AS} \) measure, the elasticity ratio for the \( P^{FH} \) measure only has two components, which correspond to the second and third factors in the \( P^{AS} \) case. In particular, now we lose the first factor \( \frac{1}{k} \), which serves as a depreciating component in the \( P^{AS} \) case. Therefore, the force driving up the importance of higher moments is even stronger in the performance evaluation based on the \( P^{FH} \) measure.

Up till now, we have finished the magnitude effect analysis with respect to raw moments. Similar results can be obtained with respect to central moments. For brevity, we do not show these results here, but they will be available upon request.

### 4.3 Rare Disasters

In some cases gambles feature very bad events that occur with a very small probability. As discussed in Section 2, small probability events are not likely to affect low moments, but may become dominant when high moments are taken into account. Thus, performance evaluation measures such as the Sharpe ratio are unable to account for rare disasters, whereas the measures discussed here are better situated to reflect such events. In fact, the two measures differ in the way they account for such events.

An important property of the \( P^{FH} \) measure is that it is extremely sensitive to rare disasters. To formalize this property we need to formally define a “rare disaster.” Let \( g_0 \in \mathcal{G} \) be a gamble and choose \( L > 0 \) very large. One can think of \( g_0 \) as a “business as usual” gamble that involves some gains and losses but no disastrous events, whereas \(-L\) is a very big and unusual loss. Then, consider the composite gamble \( g_\alpha \) that assigns probability \( 1 - \alpha \) to \( g_0 \) and \( \alpha \) to \(-L\), where \( \alpha \) is some small probability. The gamble \( g_\alpha \) reflects both “business-as-usual” realizations and the rare disaster. As \( \alpha \) becomes very small, the \( P^{FH} \) measure becomes completely dominated by the disastrous loss \( L \). Formally, \( \lim_{\alpha \to 0} P^{FH}(g_\alpha) = 1/L \). Formally,

**Theorem 5** Let \( g_0 \in \mathcal{G} \) be a gamble and \( L > 0 \) such that \( P^{FH}(g_0) > 1/L \). Let \( \alpha \in (0, 1) \) and let \( g_\alpha \) denote a composite gamble that assigns probability \( 1 - \alpha \) to \( g_0 \) and \( \alpha \) to \(-L\). Then, \( \lim_{\alpha \to 0} P^{FH}(g_\alpha) = 1/L \).

Recall that the \( P^{FH} \) measure can be interpreted as the reciprocal of the wealth level at which a log investor would be indifferent between investing in the gamble or staying with the status-quo. Thus, an interpretation of this result is that with

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\(^{5}\)It is implicit in the statement of the theorem when writing “\( \lim \)” that the associated limit exists. Note also that since \( g_0 \in \mathcal{G} \), we have that \( E(g_0) > 0 \). It follows that for all \( \alpha \) in a right neighborhood of 0, \( E(g_\alpha) > 0 \) and thereby \( g_\alpha \in \mathcal{G} \) and \( P^{FH}(g_\alpha) \) is well defined.
rare disasters, the wealth level needed is equal to the worst-case loss. It is important to note that the above theorem does not apply to the $P^{\text{AS}}$ measure. In fact, the continuity property in Aumann and Serrano (2008, page 819) implies that when $\{g_\alpha\}$ is uniformly bounded, we have that $\lim_{\alpha \to 0} P^{\text{AS}}(g_\alpha) = P^{\text{AS}}(g_0)$. That is, a rare disaster has a negligible effect on the $P^{\text{AS}}$ measure similar to standard performance evaluation measures. In Section 5 we will illustrate these properties. We will show that indeed the $P^{\text{FH}}$ measure is much more sensitive to isolated bad events than the $P^{\text{AS}}$ measure. In fact, Theorem 5 will become very useful when analyzing the performance of popular investment strategies.

4.4 Scale and Leverage

In applying the two measures in practice one should use caution when dealing with the scale of the gamble and with leverage. To see this point consider a gamble $g$, and scale it up to $\alpha g$ with $\alpha > 1$.

The homogeneity of the measures implies that $P(\alpha g) = \frac{1}{\alpha} P(g) < P(g)$. This is a simple reflection of the fact that if $g$ is rejected by a risk averse individual compared to the status quo, then $\alpha g$ must also be rejected. Indeed, fix any increasing and concave utility $u$. Then, by Jensen’s inequality

$$
\mathbf{E} u(w_0 + g) - u(w_0) \geq \frac{1}{\alpha - 1} (\mathbf{E} u(w_0 + \alpha g) - \mathbf{E} u(w_0 + g)).
$$

Hence, if $g$ is rejected (i.e., $\mathbf{E} u(w_0 + g) < u(w_0)$), then also $\alpha g$ is rejected (since $\mathbf{E} u(w_0 + \alpha g) < \mathbf{E} u(w_0 + g) < u(w_0)$). In words, scaling up a rejected gamble cannot be acceptable.

Similarly, if there exists a risk free asset with return $r > 0$, we can consider a gamble $\alpha g + (1 - \alpha) r$, assuming it is in $\mathcal{G}$. Then for $\alpha > 1$,

$$
P(\alpha g + (1 - \alpha) r) \leq P(g).
$$

This again reflects that any time a risk averse individual rejects a gamble $g$ (compared to the status quo) he also rejects a levered version of that gamble. Indeed, we already know that $P(\alpha g) < P(g)$. Then, (10) follows immediately from the fact that $\alpha g + (1 - \alpha) r$ is first-order stochastically dominated by $\alpha g$. In fact, even more can be said. Any time a gamble $g$ is rejected relative to investment in the risk free asset, levering the gamble is also rejected.

The key for practical applications of the measures is that the choice of both scale and leverage are utility dependent. However, the approach taken here is that the

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6 Any time we use $P$ it means that the statement applies to both $P^{\text{AS}}$ and $P^{\text{FH}}$.

7 To see this, note that by Jensen’s inequality

$$
\mathbf{E} u(w_0 + g) \geq \frac{\mathbf{E} u(w_0 + \alpha g + (1 - \alpha) r) + (\alpha - 1) u(w_0 + r)}{\alpha},
$$

which can be rearranged as

$$
\mathbf{E} u(w_0 + g) - u(w_0 + r) \geq \frac{1}{\alpha - 1} (\mathbf{E} u(w_0 + \alpha g + (1 - \alpha) r) - \mathbf{E} u(w_0 + g)).
$$
actual utility of a particular investor is not known. Thus, for a “fair” comparison using the performance measures discussed in this paper, gambles should have the same scale and leverage. For example, comparing the performance of two equity mutual funds using the measures is reasonable. However, to compare the performance of an equity mutual fund with a balanced fund one should first lever up the balanced fund returns, and only then compare them to those of the equity mutual fund. In the implications illustrated in the next section we make sure that all gambles considered have the same scale and leverage.

5 Implications

Having established the high-moment properties of the measures, we now illustrate several applications. We first discuss how the measures can be estimated empirically, and then provide several settings where the measures are used to draw conclusions about the attractiveness of different investment strategies. The high moment and rare event properties play an important role in these analyses.

5.1 Estimation Method

Before we turn to some applications of the performance measures, we first discuss how they can be estimated empirically from standard return data.

5.1.1 Using Return Data

In most applications the theoretical notion of a “gamble” will be represented by a finite sample of $T$ return observations. These returns reflect the percentage change in the investment of the agent. To be consistent we assume that all investments are of $100, which allows us to treat the percentage rates of return as the actual wealth changes.

Additionally, note that the performance measures do not explicitly account for the opportunity cost of investing in a particular gamble. Thus, in each case we need to specify what we view is the right opportunity cost. For example, in the case of investing in the security market one can view the opportunity cost as investing in a risk free asset. In that case, the gamble that should be evaluated is that of borrowing $100 at the risk free rate and investing this amount in the market. Thus, the returns to be evaluated are, in fact, the excess returns (returns less risk free rate). Different cases we analyze below will be associated with different natural opportunity costs. Note that by explicitly specifying the opportunity cost we also account for the time value of money which is ignored if one takes the performance measures at face value.

5.1.2 Statistical Estimation of the Measures

Consider a finite sample of $T$ return observations. As usual in the performance evaluation literature, we assign each observation a probability of $\frac{1}{T}$. Then, we estimate $P^{AS}$ and $P^{FH}$ by solving the implicit equations (1) and (3). Typically, we view the
finite sample as a random sample from a population of returns. Then, $P^{AS}$ and $P^{FH}$ which we calculate are just the sample estimates of the “true” performance measures. To answer questions regarding the “true” population performance measures we need to obtain standard errors. To do this we apply the Generalized Method of Moments (GMM) (see Hansen (1982)) in our estimation of the $P^{AS}$ and $P^{FH}$ measures. This approach is natural since (1) and (3) can directly be viewed as moment conditions, and lend themselves easily to the GMM approach. Notice that in either case, we have exactly one parameter to be estimated and one moment condition, implying that our problem is just identified, and the GMM estimate is determined such that the sample average of the moment condition equals zero. Thus, the GMM estimates equal the solutions to the implicit equations (1) and (3). These estimates are consistent and asymptotically normal. Moreover, since the model is just identified the weighting matrix used for the GMM estimation is irrelevant, and the estimates are efficient. We also obtain a covariance matrix, which yields standard errors that allow us to examine the statistical significance of the measures.

Often we compare the performance measures for multiple gambles, to assess which investment strategy dominates. We then estimate the performance measures jointly. That is, we estimate the performance measures of $n$ gambles using the $n$ moment conditions implied by the implicit equations (1) (or (3)). The resulting covariance matrix allows us to compute standard error for the difference in the measures of any two gambles, which can then be used to determine whether the gambles generate significantly different performance measures in the population.

To be consistent with the estimation of the performance measures, we also use the GMM approach to estimate various moments of gambles. These estimates are again consistent and efficient. It is worth noting that the GMM estimates of central moments are biased, but this has negligible effects on our results due to the consistency of the estimates.

5.2 Implication I: Attractiveness of Anomalies

As a first implication for the measures we evaluate the attractiveness of popular investment strategies that rely on well documented anomalies. It is well established that small firms (those with low market capitalizations), and value firms (those with high book-to-market ratios) gain abnormal average returns in US equity markets (see Banz (1981) and Rosenberg, Reid and Lanstein (1985)). Additionally, it is established that momentum strategies, i.e., holding long positions in stocks that yielded high returns in the recent past while holding a short position in stocks that yielded low returns in the recent past, generate abnormal returns (see Jegadeesh and Titman (1993)). Are these trading strategies still attractive when accounting for their high moment properties?

To evaluate this issue we us the Fama and French (1993) and Carhart (1997) portfolios, which are constructed based on these anomalies. Specifically, Fama and French (1993) and Carhart (1997) construct four portfolios. The first is denoted $mktrf$ (for market less the risk free rate). The returns for this portfolio reflect an investment in a well diversified portfolio of US stocks, where the opportunity cost.
is assumed to be investment in a risk free asset. The second portfolio is $smb$ (for small minus big). This portfolio is long in low market-capitalization stocks and short in high market-capitalization stocks. Historically, this portfolio yielded abnormal returns reflecting the “small firm” anomaly. Note that the underlying assumption here is that the opportunity cost of investing in small stocks is the return of investing in large stocks. The third portfolio is $hml$ (for high minus low). This portfolio is long in high book-to-market stocks (value stocks) and short in low book-to-market stocks (growth stocks). This portfolio builds on the “value anomaly.” The fourth portfolio is $umd$ (for up minus down). This portfolio is long in stocks that had high returns in the recent past (winners) and short in stocks that had low returns in the recent past (losers), building on the momentum anomaly.

We obtain monthly data for the four portfolios for the period January 1962 to December 2009 from Kenneth French’s data library. Panel A of Table 3 reports summary statistics for the first moment ($\mu_1$) and the three higher central moments ($m_2$, $m_3$, and $m_4$) for the four portfolios. The table also compares the moments for the different portfolios using the GMM standard errors.

The average monthly market excess return ($mktrf$) during our sample period is 0.41%, the average monthly return on the $smb$ portfolio is 0.23%, and the average monthly returns on the $hml$ and $umd$ portfolios are 0.44% and 0.73%, respectively. These averages are consistent with prior studies, and reflect the popular attractiveness of the value ($hml$) and momentum ($umd$) anomalies. In terms of higher moments we see that $mktrf$ has the highest variance ($m_2$), while $umd$ is the most negatively skewed and exhibits the largest tail risk ($m_4$) out of the four portfolios. Figure 1 presents histograms of the returns of the four portfolios. One interesting feature that can be learned from this figure is that the $umd$ portfolio has some extreme and rare bad events. These “rare disasters” contribute to the high tail risk associated with this strategy and perhaps also to the left skewness.

Panel B of Table 3 reports the Sharpe ratios and the $P^{AS}$ and $P^{FH}$ performance measures. The Sharpe ratio of the $umd$ portfolio appears to be the highest among the four, with the Sharpe ratio of the $smb$ the lowest, although the differences are not statistically significant. Both the $P^{AS}$ and $P^{FH}$ measures, however, suggest that $hml$ is the superior portfolio. Apparently, the high negative skewness and tail risk associated with the momentum strategy lower its attractiveness. Indeed, comparing the $umd$ and $hml$ portfolios we see that the former has more negative skewness (-116.1 vs. -0.67) and higher tail risk (4862.9 vs. 390.8). Thus, despite the fact that the Sharpe ratio of $umd$ is slightly higher than that of $hml$, the $P^{AS}$ and $P^{FH}$ measures favor $hml$.

It is interesting to note that the negative rare events showing in the distribution of $umd$ play an important role in the determination of the $P^{FH}$ measure. Recall from Theorem 5 that in the presence of a rare disaster, we have $P^{FH} \approx \frac{1}{L}$, where $L$ is the loss in the case of the disaster. In the case of $umd$, Figure 1 shows a “disaster return” of

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8We estimate Sharpe ratios and their associated standard errors following Lo (2002). This is achieved by applying the delta method based on the GMM estimates of the mean and the variance of the portfolio returns.
-34.7%. This reflects the return on momentum investing in April 2009, during which the market experienced a sharp reversal. Accordingly, we have that \( P^{FH} \) for the \( umd \) portfolio is \( \frac{1}{\sigma^2} \approx 0.029 \). Thus, in the case of \( umd \), the rare disaster dominates the performance measurement of the portfolio, when using the \( P^{FH} \) measure.

Panel B of Table 3 also reports the elasticities, which reflect the importance of the different moments in the determination of the performance measures (as discussed in Section 4.2). Recall, in particular, that the importance of the moments may or may not be monotone. Thus, higher moments can potentially be very influential in determining performance. The elasticities reported in the table suggest that the first two moments have a very strong effect on performance of the four portfolios. The elasticities of the third and fourth moments are lower but still meaningful. And, as suggested in Section 4.2, the elasticities are often non-monotone. For example, for the \( P^{AS} \) measure of the \( hml \) portfolio, the fourth moment has a roughly 20 times larger elasticity compared to the third moment, due to the very small skewness of this portfolio. Note that in the case of the \( P^{FH} \) measure of the \( umd \) portfolio all reported elasticities are very small. This is a reflection of the fact that in this case the \( P^{FH} \) measure is dominantly determined by the rare disaster.

Finally, Panel B also reports the expected utility of a risk-averse investor with CRRA utility given by \( u(w) = \frac{w^{1-\gamma}}{1-\gamma} \). For illustration we use \( \gamma = 5 \), which is the level of risk aversion commonly used in asset pricing calibrations. In addition, we set the initial wealth and the scale of investment to be equal. The results suggest that the \( hml \) portfolio gives rise to the highest expected utility, while \( mktrf \) yields the lowest. However, the differences in the expected utility for the four portfolios are not statistically significant.

In summary, the discussion in this section illustrates that high moments and rare disasters may have a meaningful effect on the performance evaluation of popular trading strategies. In particular, the popular momentum strategy appears much less attractive using the new performance measures. Indeed, this strategy has high negative skewness, high tail risk, and it exhibits rare disasters, all of which tend to lower its \( P^{AS} \) and \( P^{FH} \) performance measures.

### 5.3 Implication II: More on the Importance of High Moments and Rare Disasters

In this section we further illustrate the importance of high moments in evaluating the performance of investment portfolios. To illustrate this point we use the momentum (\( umd \)) portfolio presented above. We ask the following question: To what extent would performance evaluation be biased if we accounted for the first two moments only, essentially ignoring the effects of higher moments? To answer this question we draw 1,000,000 realizations from a simulated normal distribution with mean and variance equal to those of the sample momentum returns. The two distributions are depicted in Figure 2. Panel A of Table 4 reports the moments of the true \( umd \) distribution and those of the simulated distribution. Observe that the first two moments are virtually identical (as expected). The higher moments look quite different. For example, the fourth moment of \( umd \) is 4863 compared to 1062 for the simulated dis-
distribution. However, the GMM standard errors are quite large, and we cannot reject the null that the two are statistically equal. Panel B of Table 4 reports the Sharpe ratio and the $P^{AS}$ and $P^{FH}$ performance measures. The Sharpe ratio is almost identical for the true and simulated portfolios, which is expected as both portfolios have similar mean and variance. The $P^{AS}$ measure is lower for the umd portfolio as compared to the simulated one (0.063 vs. 0.077), although the difference is not statistically significant. The $P^{FH}$ measure significantly favors the simulated normal distribution, as it is about 72% higher than that for the umd portfolio (significant at the 1% level). The difference in the $P^{FH}$ measures of the two distributions comes from the worst outcomes, which can be considered “rare disasters” as discussed in Section 4.3. Recall that we defined a “rare disaster” as an outcome $-L$ (where $L$ is a positive number) such that $P^{FH}$ of the gamble without $-L$ is higher than $\frac{1}{L}$. Both the true umd and the simulated portfolio exhibit rare disasters according to this definition. In the umd case it is the realization of -34.7% (corresponding to April 2009). In the simulated portfolio it is the lowest realization of -20.3% (not visible in the plot). This example again shows that the Sharpe ratio misses significant information incorporated in high moments. Note that in this particular example, the differences between the two distributions is largely incorporated in the rare disasters. Hence, while the $P^{FH}$ measure provides a sharp distinction between the cases, the $P^{AS}$ measure does not.

5.4 Implication III: Private vs. Public Equity

In our next illustration of the performance measures we compare the performance of private and public equity. Moskowitz and Vissing-Jorgensen (2002) provide a thorough comparison of the performance of public vs. private equity investment from the point of view of individual investors. They find that the returns to private equity are not higher than those of public equity. This result is puzzling since private equity investments expose investors to a high level of idiosyncratic risk. They observe that private equity investment is right skewed and conjecture that preference for skewness may be one reason for the tendency of individuals to invest in private equity.

The performance measures used in this paper account for all moments of the distribution (skewness in particular). We thus use the measures to explore whether the right skewness of private equity indeed compensates for its otherwise unattractive performance. Our method follows Moskowitz and Vissing-Jorgensen (2002). Using data on individual household investment in private equity from the 2004 Survey of Consumer Finance (SCF), we estimate excess returns obtain by households since the founding or the acquisition of a private firm. These are the returns they would achieve by borrowing at the risk free rate and investing in the private firm. We consider only private firms in which a household has its largest actively managed equity position. We treat each household as an observation and estimate the average annual holding period return. We disregard observations with a holding period of less than one year. The average annual holding return is calculated as the sum of the geometric average annual capital gain and the current dividend return, which is assumed to be representative of those in previous years. While the former can be estimated from the
initial and current market value of the ownership share, the latter is computed from
the earnings in the year prior to the survey and the current market value of equity
assuming that 30 percent of earnings are retained in the firm. Finally, we subtract
the risk free rate from the estimated holding period return to obtain the average annual
excess return. Notice that similar to Moskowitz and Vissing-Jorgensen (2002), our
sample is conditional on survival of the private firms, yielding an upward bias in the
evaluation of private equity performance.

For the purpose of comparison, we compute for each household the geometric av-
erage annual return it would obtain by investing in the CRSP value-weighted market
index for the same time period as its private equity holdings. As before, we subtract
the risk free rate to get the average excess return of public equity. Figure 3 shows the
distributions of private and public equity.\footnote{These histograms are quite similar to those reported in Figure 2 of Moskowitz and Vissing-
Jorgensen (2002) which are based on survey data from 1989 (as opposed to 2004 in our case).} For the sake of plotting the private equity
histogram we winsorize the excess returns at 200%. Importantly, we do not winsorize
the data in any of our analyses, since extreme events are key to our results. In line
with the results of Moskowitz and Vissing-Jorgensen (2002), the figures suggest that
private equity returns are much more dispersed and right skewed compared to public
equity.

Panel A of Table 5 reports the moments for the private and public equity returns.
The average return on private equity conditional on survival is about 35 times larger
than that of public equity, and the difference is significant at the 1% level. However,
private equity is also extremely more volatile. The higher moments are also very
intuitive. As expected, the third moment of private equity is much larger than that
of public equity, but it also exhibits much heavier tails as reflected in the fourth
moment. Overall, based on these four moments it is not clear a priori which of the
two dominates.

Panel B of Table 5 provides the calculations of the performance measures. The
Sharpe ratio of private equity is significantly lower than that of public equity (at the
1% level). That is, the superior average returns of private equity are outweighed by
their high volatility. The question then is whether the higher moment properties
of private equity make it attractive (as conjectured by Moskowitz and Vissing-Jorgensen
(2002)). The answer provided by the $P^{AS}$ and $P^{FH}$ measures is negative. Both
measures are significantly higher for public equity as opposed to private equity, even
when conditional on survival. The right skewness of private equity is not sufficient
to compensate for its other moment properties. In other words, the "private equity
premium puzzle" suggested by Moskowitz and Vissing-Jorgensen (2002) does not
seem to be resolved by high moment properties.

5.5 Implication IV: Active vs. Passive Funds

In the next illustration of the measures we compare the performance of actively
managed equity funds and index equity funds. There is a long lasting debate on the
value of active as opposed to passive management of mutual funds (see for example
Jensen (1968), Henriksson (1984), Chang and Lewellen (1984), Ippolito (1989), and Malkiel (1995)). One aspect of this debate that has not received much attention is the high moment properties of the portfolios selected by active mutual funds when compared to index funds. To examine this issue we obtain mutual fund return data from CRSP for the period January 1991 to December 2009. This dataset includes the past records of all open-ended mutual funds in existence during the sample period and is thus free of survivorship bias. The returns we use are net of fees.

To evaluate the performance of actively managed funds, we drop all index funds (including index-based and enhanced index funds), bond funds and balanced funds and keep only the actively managed equity funds. Also excluded from our analysis are funds investing in foreign securities and sector funds that invest in particular industries, such as utilities and real estate. Similarly, we obtain a sample of index equity funds by dropping all actively managed funds, bond funds, balanced funds, foreign funds and sector funds.\footnote{In our sample of index fund we do not include index-based and enhanced index funds. Including those in the analysis does not change the results materially and the conclusions are unaffected.} We then compute for each month during the sample period the value weighted average return of all actively managed funds and index funds, and we subtract the risk free rate to obtain excess returns (228 observations overall). The distributions of excess returns for the two groups are plotted in Figure 4. From the figure, the two distributions look quite similar.

Panel A of Table 6 reports summary statistics for both groups. It can be seen that all four moments are very similar for the active and passive funds, and the differences between them are not statistically significant. Panel B reports the Sharpe ratio and the $PAS$ and $PFH$ measures. Here again, we do not see significant differences across the two fund groups, consistent with the similar moments reported in Panel A. Thus, the $PAS$ and $PFH$ measures lead to the conclusion that after-fee returns of active and index funds are not distinguishable, even after accounting for high moments.

## 6 Mutual Fund Selection Based on the Performance Measures

So far we have illustrated how the new measures can be used to evaluate the performance of investments. In this section we take one step forward and examine the performance of investments based on the new measures. To this end, we use the new performance measures to select mutual funds on a monthly basis, and then examine the outcome of such an investment strategy.

Our data for this exercise consists of monthly returns of all mutual funds provided by the Center for Research in Security Prices (CRSP), and the Mutual Fund Links (MFLinks) data maintained by the Wharton Research Data Services (WRDS). We focus on a sample period that covers 576 months from January 1962 through December 2009.

The mutual fund return data from CRSP include the past records of all open-ended mutual funds in existence during our sample period and is free of survivorship bias. The original dataset covers 44093 funds. We drop all index funds, bond funds
and balanced funds and keep only the actively managed equity funds. We also exclude from our analysis foreign funds and sector funds. This leaves us with 15580 actively managed, well-diversified mutual funds that invest in domestic equity securities.

The MFLinks data provide a reliable means to join the CRSP Mutual Fund data to equity holdings information in the TFN/CDA S12 datasets. In practice, an investment company may run different portfolios that have the same composition but serve different groups of investors. Such portfolios are treated as separate mutual funds in CRSP as noted in Wermers (2000). The MFLinks allows us to combine all these portfolios, yielding 3222 unique funds, which are to be the primary objects of our empirical study.

Our aim is to examine the performance of mutual portfolios selected based on the new measures as compared to those selected based on a traditional measure (the Sharpe ratio). In each month during our sample period starting from January 1967 and for each mutual fund we calculate the $P^{AS}$ and $P^{FH}$ measures based on the most recent 60 monthly excess returns assigning equal probability (1/60) to each observation. If a fund does not have complete records for the preceding 60 months, we disregard it for that particular month. We then rank all mutual funds in each month based on their measures (separately for $P^{AS}$ and $P^{FH}$). We then obtain two portfolios of “selected” mutual funds by constructing an equal-weighted portfolio of the top decile mutual funds dropping any mutual funds for which the measures cannot be calculated.\footnote{Recall that the measures can be calculated only if the average return is positive. In some cases following very bad years this prevents us from calculating the measures for some funds over a 60-month period.} We rebalance these portfolios on a monthly basis. For the purpose of comparison, we construct an investment strategy based on the Sharpe ratio (denoted by $S$) of the funds in a similar manner, excluding funds for which the Sharpe ratio is negative.

Panel A of Table 7 compares various moments and performance measures for the portfolio excess returns obtained by adopting the three portfolios described above (denoted for brevity by AS, FH, and S), as well as for the market (denoted by MKT, and represented by the CRSP value-weighted index). For each of the four portfolios the table reports $\mu_1$, $m_2$, $m_3$ and $m_4$, which represent the 1\textsuperscript{st} moment and the 2\textsuperscript{nd}, 3\textsuperscript{rd} and 4\textsuperscript{th} central moments estimated using GMM.

The average return ($\mu_1$) appears highest for portfolio S (0.54\%), which was constructed based on Sharpe ratios. Indeed, the average returns on the AS and FH portfolios are estimated at 0.50\% and 0.47\% respectively, and the average return on the market is estimated at 0.43\%. However, the differences are insignificant in most cases. By contrast, higher moments generated by the AS and FH measures are significantly more appealing than those generated by the Sharpe ratio and are also often more appealing than those generated by the market. Consider first the second central moment ($m_2$). The estimated values for the AS and FH portfolios are 16.2 and 15.7 respectively. Those are significantly lower than the estimates for the S and MKT portfolios, which are 21.2 and 21.4, respectively. Now, consider the third central moment ($m_3$). Its estimates for the AS and FH portfolios are -37.3 and -37.0
respectively as opposed to -59.1 and -55.6 for the S and market portfolios respectively, with the difference being significant when compared to the S portfolio. Thus, the AS and FH measures appear to generate less negatively skewed returns compared to the S portfolio. Finally, both the AS and the FH portfolios generate a significantly lower fourth central moment \( (m_4) \) compared to both the S and MKT portfolios. Thus, the AS and FH portfolios seem to present investors with a lower tail risk when compared to the market or the Sharpe ratio based portfolios.

The two performance measures estimated for the four different portfolios encapsulate these observations. Indeed, both \( P^{AS} \) and \( P^{FH} \) are higher for the AS and FH portfolios compared to S and MKT portfolios, where the difference is statistically significant in most cases. Interestingly, portfolio S, which is composed based on the Sharpe ratio has a lower Sharpe ratio compared to the AS and FH portfolios, though the differences are not statistically significant.

To further illustrate the performance of the portfolios in the eyes of a risk-averse investor, in Panel B of Table 7 we calculate the expected utility obtained from investing in each one of these portfolios. We assume the initial wealth level and the scale of investment to be equal. We also assume the investor possesses a CRRA utility \( u(w) = \frac{w^{1-\gamma}}{1-\gamma} \) with the risk-aversion coefficient, \( \gamma \), taking the values 3,5,10,15, and 20 (these values are quite typical in asset pricing calibration exercises). Obviously, the utility values themselves have no meaning since Von-Neumann-Morgenstern utilities can always be rescaled. The focus of this table is on the differences in utilities generated by the different portfolios. The statistical significance of these differences is independent of scale. As can be seen from the table, the average utility generated by the AS and FH portfolios is higher than the expected utility generated by the S portfolio and the market. The differences become more significant as the risk aversion coefficient becomes larger.

7 Conclusion

To be completed.

Appendix

Proof of Theorem 3: To save space, we only provide the proof for the first part of the Theorem and for the case that \( k \) is an odd number. The proof of all the other cases is similar. For all non-negative real numbers \( p \geq 0 \) and all gambles \( g \in \mathcal{G} \), define

\[
f^{AS}(p, g) \equiv E[\exp(-pg)] - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} p^n \mu_n(g). \tag{11}
\]

By Theorem 1 we know that \( f^{AS}(\cdot, g) \) crosses zero at \( p = P^{AS}(g) > 0 \). Moreover, from the fact that \( \mu_1(g) > 0 \), it follows that

\[
\frac{\partial f^{AS}(0, g)}{\partial p} = -\mu_1(g) < 0. \tag{12}
\]
Also, for all $p \geq 0$

$$\frac{\partial^2 f^{AS}(p, g)}{\partial p^2} = \mathbb{E}[\exp(-pg)g^2] > 0,$$

(13)

implying that $f^{AS}(\cdot, g)$ is strictly convex. From (12) and (13) and using that fact that $f^{AS}(0, g) = 0$, it follows that $f^{AS}(\cdot, g)$ crosses zero at $p = P^{AS}(g) > 0$ from below.

Now, suppose that $g, g' \in \mathcal{G}$ such that $\mu_k(g) > \mu_k(g')$. Since $k$ is odd, the coefficient of $\mu_k(g)$ in $f^{AS}(p, g)$ is negative. Hence,

$$f^{AS}(p, g) < f^{AS}(p, g')$$

(14)

for all $p > 0$. By Eq. (5) we know that

$$f^{AS}(P^{AS}(g), g) = 0.$$ 

(15)

Then, by (14) and (15) we have

$$0 = f^{AS}(P^{AS}(g), g) < f^{AS}(P^{AS}(g), g').$$

(16)

But, as we have shown, $f^{AS}(\cdot, g')$ crosses zero at $P^{AS}(g')$ from below (and does not hit zero to the right of $P^{AS}(g')$ due to the convexity of $f^{AS}(\cdot, g)$). Hence, it must be that $P^{AS}(g') < P^{AS}(g)$.

Proof of Theorem 4: To save space, we only provide the proof for the first part of the Theorem and for the case that $k$ is an odd number. The proof of all the other cases is similar. For all non-negative real numbers $p \geq 0$ and all gambles $g \in \mathcal{G}$, define

$$f^{FH}(p, g) \equiv \mathbb{E}[\log(1 + pg)] = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} p^n \mu_n(g).$$

(17)

By Theorem 2 we know that $f^{FH}(\cdot, g)$ crosses zero at $p = P^{FH}(g) > 0$. Moreover, from the fact that $\mu_1(g) > 0$, it follows that

$$\frac{\partial f^{FH}(0, g)}{\partial p} = \mu_1(g) > 0.$$ 

(18)

Also, for all $p \geq 0$

$$\frac{\partial^2 f^{FH}(p, g)}{\partial p^2} = -\mathbb{E}[g^2/(1 + pg)^2] < 0,$$

(19)

implying that $f^{FH}(\cdot, g)$ is strictly concave. From (18) and (19) and using that fact that $f^{FH}(0, g) = 0$, it follows that $f^{FH}(\cdot, g)$ crosses zero at $p = P^{FH}(g) > 0$ from above.

Now, suppose that $g, g' \in \mathcal{G}$ such that $\mu_k(g) > \mu_k(g')$. Since $k$ is odd, the coefficient of $\mu_k(g)$ in $f^{AS}(p, g)$ is positive. Hence,

$$f^{FH}(p, g) < f^{FH}(p, g')$$

(20)
for all $p > 0$. By Eq. (5) we know that

$$f^{FH}(P^{FH}(g), g) = 0. \tag{21}$$

Then, by (20) and (21) we have

$$0 = f^{FH}(P^{FH}(g), g) > f^{FH}(P^{FH}(g), g'). \tag{22}$$

But, as we have shown, $f^{FH}(\cdot, g')$ crosses zero at $P^{FH}(g')$ from above (and does not hit zero to the right of $P^{FH}(g')$ due to the concavity of $f^{FH}(\cdot, g)$). Hence, it must be that $P^{FH}(g') < P^{FH}(g)$. \[\blacksquare\]

**Proof of Theorem 5:** Denote $P^{FH}_\alpha \equiv P^{FH}(g_\alpha)$ for $\alpha \in [0, 1)$. We first show that $P^{FH}_\alpha$ converges as $\alpha$ goes to zero. To see this, note that $\forall \alpha \in A$,

$$0 = E[\log(1 + P^{FH}_\alpha g_\alpha)] = (1 - \alpha) E[\log(1 + P^{FH}_\alpha g_0)] + \alpha \log(1 - P^{FH}_\alpha L). \tag{23}$$

By implicitly differentiating (23), one can verify that $\frac{\partial P^{FH}_\alpha}{\partial \alpha} < 0.\footnote{To verify this point one should recall that at $P^{FH}$ the function $f^{FH}$ crosses zero from above (see the proof of Theorem 4).}$ That is, $P^{FH}_\alpha$ is monotonically decreasing in $\alpha$. Since by definition $0 < P^{FH}_0 < 1/L$, we know that $P^{FH}_\alpha$ is monotonically decreasing and bounded. Hence, $\lim_{\alpha \to 0} P^{FH}_\alpha$ exists.

Since $P^{FH}_\alpha > 0$ and $L > 0$, we have that $\log(1 - P^{FH}_\alpha L) \neq 0$. Therefore, we can rewrite (23) as

$$\frac{E[\log(1 + P^{FH}_\alpha g_0)]}{\log(1 - P^{FH}_\alpha L)} = \frac{\alpha}{1 - \alpha},$$

which implies that

$$\lim_{\alpha \to 0} \frac{E[\log(1 + P^{FH}_\alpha g_0)]}{\log(1 - P^{FH}_\alpha L)} = \lim_{\alpha \to 0} \frac{\alpha}{1 - \alpha} = 0. \tag{24}$$

Since $P^{FH}_\alpha$ converges and $0 < P^{FH}_\alpha < 1/L$ for all $\alpha \in A$, there are three possible cases: (i) $0 < \lim_{\alpha \to 0} P^{FH}_\alpha < 1/L$; (ii) $\lim_{\alpha \to 0} P^{FH}_\alpha (g_\alpha) = 0$; and (iii) $\lim_{\alpha \to 0} P^{FH}_\alpha (g_\alpha) = 1/L$. To show the result, it is thus sufficient to show that cases (i) and (ii) yield a contradiction.

We consider each of these two cases separately.

Case (i): If $0 < \lim_{\alpha \to 0} P^{FH}_\alpha (g_\alpha) < 1/L$, then

$$-\infty < \lim_{\alpha \to 0} \log(1 - P^{FH}_\alpha L) < 0.$$ 

In addition, by the fact that $E[\log(1 + P^{FH}_0 g_0)] = 0$ and $P^{FH}_0 > 1/L$, we are guaranteed that

$$\lim_{\alpha \to 0} E[\log(1 + P^{FH}_\alpha g_0)] = 0.$$ 

Thus, we have

$$\lim_{\alpha \to 0} \frac{E[\log(1 + P^{FH}_\alpha g_0)]}{\log(1 - P^{FH}_\alpha L)} \neq 0,$$
which yields a contradiction to (24).

Case (ii): If \( \lim_{\alpha \to 0} P_{FH}^{\alpha} (g_0) = 0 \), then

\[
\begin{align*}
  \lim_{\alpha \to 0} & \frac{E[\log(1 + P_{FH}^{\alpha} g_0)]}{\log(1 - P_{FH}^{\alpha} L)} \\
  &= \lim_{\alpha \to 0} \frac{E\left[ g_0 \frac{\partial P_{FH}^{\alpha}}{\partial \alpha} \right]}{1 + P_{FH}^{\alpha} g_0} \\
  &= \lim_{\alpha \to 0} \frac{-L}{1 - P_{FH}^{\alpha} L} \\
  &= - \frac{E[g_0]}{L} \\
  &< 0,
\end{align*}
\]

where the first equality follows from L'Hopital’s rule, and the second equality results from the fact that \( \frac{\partial P_{FH}^{\alpha}}{\partial \alpha} < 0 \). Hence, this case is also ruled out. ■

References


Table 1: Motivating Example I

This table shows an example in which the Sharpe ratio fails to capture the high moment and rare disaster properties of the distribution of gambles when first-order stochastic dominance works. Two gambles \( g_1 \) and \( g_2 \) are being compared, with the gamble value and the corresponding probability and \( CDF \) reported. The table also reports the various moments of the two gambles, where \( \mu_1 \) denotes the first moment, and \( m_2, m_3, \) and \( m_4 \) stand for the second, third and fourth central moments, respectively. In addition, the table also reports the Sharpe ratios of the gambles, denoted by \( S \).

<table>
<thead>
<tr>
<th></th>
<th>Value</th>
<th>Probability</th>
<th>( CDF )</th>
<th>Value</th>
<th>Probability</th>
<th>( CDF )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g_1 )</td>
<td></td>
<td></td>
<td></td>
<td>( g_2 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-10</td>
<td>0.001</td>
<td>0.001</td>
<td></td>
<td>-1</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>1</td>
<td>0.999</td>
<td>1</td>
<td></td>
<td>1</td>
<td>0.9</td>
<td>0.901</td>
</tr>
<tr>
<td>4</td>
<td>0.099</td>
<td>1</td>
<td></td>
<td>4</td>
<td>0.099</td>
<td>1</td>
</tr>
<tr>
<td>( \mu_1 )</td>
<td>0.989</td>
<td></td>
<td></td>
<td></td>
<td>1.295</td>
<td></td>
</tr>
<tr>
<td>( m_2 )</td>
<td>0.121</td>
<td></td>
<td></td>
<td></td>
<td>0.808</td>
<td></td>
</tr>
<tr>
<td>( m_3 )</td>
<td>-1.327</td>
<td></td>
<td></td>
<td></td>
<td>1.924</td>
<td></td>
</tr>
<tr>
<td>( m_4 )</td>
<td>14.583</td>
<td></td>
<td></td>
<td></td>
<td>5.335</td>
<td></td>
</tr>
<tr>
<td>( S )</td>
<td>2.845</td>
<td></td>
<td></td>
<td></td>
<td>1.441</td>
<td></td>
</tr>
</tbody>
</table>
Table 2: Motivating Example II

This table shows an example in which the Sharpe ratio fails to capture the high moment and rare disaster properties of the distribution of gambles when first-order stochastic dominance does not work. Two gambles \( g_1 \) and \( g_2 \) are being compared, with the gamble value and the corresponding probability and CDF reported. The table also reports the various moments of the two gambles, where \( \mu_1 \) denotes the first moment, and \( m_2, m_3, \) and \( m_4 \) stand for the second, third and fourth central moments, respectively. In addition, the table also reports the Sharpe ratios of the gambles, denoted by \( S \).

<table>
<thead>
<tr>
<th></th>
<th>( g_1 )</th>
<th></th>
<th>( g_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>Probability</td>
<td>CDF</td>
<td>Value</td>
</tr>
<tr>
<td>-8</td>
<td>0.001</td>
<td>0.001</td>
<td>-0.6</td>
</tr>
<tr>
<td>4</td>
<td>0.999</td>
<td>1</td>
<td>0.2</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\mu_1 & = 3.988 & \mu_2 & = 0.144 \\
m_2 & = 0.144 & m_3 & = -1.723 \\
m_3 & = -1.723 & m_4 & = 20.653 \\
m_4 & = 20.653 & S & = 10.515 \\
S & = 10.515 & & = 7.878
\end{align*}
\]
Table 3: Attractiveness of Anomalies

This table compares the attractiveness of investments in the market excess return (\(mktrf\)), and the small-minus-big (\(smb\)), high-minus-low (\(hml\)) and up-minus-down (\(umd\)) portfolios, based on the monthly returns in percentage points for the period from January 1962 to December 2009. Columns (1)-(4) of Panel A report the GMM estimates of various moments of the portfolio returns, where \(\mu_1\) denotes the first moment, and \(m_2\), \(m_3\), and \(m_4\) stand for the second, third and fourth central moments, respectively. Columns (5)-(10) report differences in the moments from investing in the various portfolios. The t-statistics are reported in the parentheses below the corresponding estimates. Asterisks denote statistical significance at the 1% (***) 5% (**) and 10% (*) levels.

<table>
<thead>
<tr>
<th>Panel A: Various Moments</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
<th>(7)</th>
<th>(8)</th>
<th>(9)</th>
<th>(10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mu_1)</td>
<td>0.4059</td>
<td>0.2298</td>
<td>0.4391</td>
<td>0.7257</td>
<td>0.1761</td>
<td>-0.0332</td>
<td>-0.3198</td>
<td>-0.2093</td>
<td>-0.4959</td>
<td>-0.2866</td>
</tr>
<tr>
<td>(2.15)**</td>
<td>(1.75)*</td>
<td>(3.62)**</td>
<td>(4.02)**</td>
<td>(0.91)</td>
<td>(-0.13)</td>
<td>(-1.14)</td>
<td>(-1.05)</td>
<td>(-2.20)**</td>
<td>(-1.23)</td>
<td></td>
</tr>
<tr>
<td>(12.02)***</td>
<td>(8.74)***</td>
<td>(11.42)***</td>
<td>(6.71)***</td>
<td>(0.57)</td>
<td>(1.44)</td>
<td>(-3.28)***</td>
<td>(-3.85)***</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(m_3)</td>
<td>-50.9774</td>
<td>16.6665</td>
<td>-0.6708</td>
<td>-116.1113</td>
<td>65.1339</td>
<td>17.3373</td>
<td>132.7778</td>
<td>115.4405</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(-2.02)**</td>
<td>(0.87)</td>
<td>(-0.11)</td>
<td>(-1.46)</td>
<td>(-2.19)**</td>
<td>(0.77)</td>
<td>(0.76)</td>
<td>(1.66)*</td>
<td>(1.44)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(m_4)</td>
<td>2092.19</td>
<td>850.61</td>
<td>390.76</td>
<td>4862.86</td>
<td>1241.58</td>
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<tr>
<td>(3.45)***</td>
<td>(2.03)**</td>
<td>(4.27)***</td>
<td>(1.73)*</td>
<td>(1.71)*</td>
<td>(2.81)**</td>
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<td>(1.21)</td>
<td>(-1.42)</td>
<td>(-1.59)</td>
<td></td>
</tr>
</tbody>
</table>
Panel B: Performance Measures and Utility Comparison

Panel B reports the Sharpe ratio ($S$) and the Aumann-Serrano and Foster-Hart performance measures ($P_{AS}$ and $P_{FH}$) for different investment portfolios in columns (1)-(4). The difference between the various portfolios is reported in columns (5)-(10). This panel also reports the absolute elasticities of the $P_{AS}$ ($P_{FH}$) measure with respect to the standardized moments, denoted as $\text{AS}_1$, $\text{AS}_2$, $\text{AS}_3$, and $\text{AS}_4$ ($\text{FH}_1$, $\text{FH}_2$, $\text{FH}_3$, and $\text{FH}_4$). The absolute elasticities of the $P_{FH}$ measure cannot be computed because $-1 < P_{FH}(g) \leq 1$ does not hold and hence the Taylor's expansion needed for the calculation of the absolute elasticities does not converge.

Further, this panel reports the expected utility ($u$) of a risk-averse investor with CRRA utility and a risk aversion coefficient of 5, assuming that the initial wealth and the investment scale are equal. The t-statistics are reported in the parentheses below the corresponding estimates. Asterisks denote statistical significance at the 1% (**), 5% (*), and 10% (**) levels.

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
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<th>(7)</th>
<th>(8)</th>
<th>(9)</th>
<th>(10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td>0.0382</td>
<td>0.0468</td>
<td>0.1014</td>
<td>0.0630</td>
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<td>0.0384</td>
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<tr>
<td></td>
<td>(2.08)**</td>
<td>(1.74)**</td>
<td>(3.15)***</td>
<td>(3.15)***</td>
<td>(-0.32)</td>
<td>(-1.13)</td>
<td>(-0.86)</td>
<td>(-1.22)</td>
<td>(-0.47)</td>
<td>(1.03)</td>
</tr>
<tr>
<td>$P_{AS}$</td>
<td>0.0083</td>
<td>0.0063</td>
<td>0.0077</td>
<td>0.0072</td>
<td>-0.0006</td>
<td>-0.0034</td>
<td>-0.0019</td>
<td>-0.0035</td>
<td>-0.0002</td>
<td>0.0009</td>
</tr>
<tr>
<td></td>
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<td>(3.16)***</td>
<td>(3.16)***</td>
<td>(-0.32)</td>
<td>(-1.13)</td>
<td>(-0.86)</td>
<td>(-1.22)</td>
<td>(-0.47)</td>
<td>(1.03)</td>
</tr>
</tbody>
</table>
| $\text{AS}_1$ | 0.9602 | 1.0081 | 0.9638 | 0.7716 | & $\text{AS}_2$ | 1.8263 | 2.0320 | 1.8101 | 1.2041 | & $\text{AS}_3$ | 0.0869 | 0.0795 | 0.0072 | 0.2345 | & $\text{AS}_4$ | 0.0454 | 0.0634 | 0.1427 | 0.2063 | & $\text{AS}_5$ | 0.0347 | 0.0449 | 0.0800 | 0.0288 | & $\text{AS}_6$ | 0.0101 | 0.0059 | 0.0059 | 0.0009 | & $\text{AS}_7$ | 0.0015 | 0.0015 | 0.0015 | 0.0015 | & $\text{AS}_8$ | 0.0009 | 0.0009 | 0.0009 | 0.0009 | & $\text{AS}_9$ | 0.0009 | 0.0009 | 0.0009 | 0.0009 | & $\text{AS}_{10}$ | 0.0009 | 0.0009 | 0.0009 | 0.0009 | & $\text{AS}_{11}$ | 0.0009 | 0.0009 | 0.0009 | 0.0009

| $u$   | -0.2514 | -0.2502 | -0.2478 | -0.2486 | -0.0013 | -0.0037 | -0.0013 | -0.0013 | -0.0013 | 0.0009 |
|       | (-116.83)*** | (-116.83)*** | (-200.81)*** | (-89.71)*** | (-0.60) | (-1.33) | (-0.76) | (-1.19) | (-0.49) | (0.27) |
Table 4: High Moment and Rare Disaster Properties of Momentum Portfolio

This table illustrates the high moment and rare disaster properties of the momentum \((umd)\) portfolio, based on the monthly \(umd\) returns in percentage points for the period from January 1962 to December 2009. The results for the \(umd\) portfolio are reported in Column (1). In comparison, 1,000,000 realizations are drawn from a simulated normal distribution with mean and variance equal to those of the sample momentum returns. The results for the simulated distribution are reported in Column (2). In addition, the difference between the true and the simulated distributions is reported in Column (3). Panel A shows the GMM estimates of various moments of the portfolio returns, where \(\mu_1\) denotes the first moment, and \(m_2\), \(m_3\), and \(m_4\) stand for the second, third and fourth central moments, respectively. Panel B reports the Sharpe ratio \((S)\) and the Aumann-Serrano and Foster-Hart performance measures \((P_{AS} \text{ and } P_{FH})\). Further, Panel B also reports the expected utility \((u)\) of a risk-averse investor with CRRA utility and a risk aversion coefficient of 5, assuming that the initial wealth and the investment scale are equal. The t-statistics are reported in the parentheses below the corresponding estimates. Asterisks denote statistical significance at the 1% (***) , 5% (**) and 10% (*) levels.

<table>
<thead>
<tr>
<th>Panel A: Various Moments</th>
<th>(\mu_1)</th>
<th>(m_2)</th>
<th>(m_3)</th>
<th>(m_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{umd})</td>
<td>0.7257</td>
<td>18.7827</td>
<td>-116.1113</td>
<td>4862.86</td>
</tr>
<tr>
<td>Simulated</td>
<td>0.7259</td>
<td>18.8181</td>
<td>0.2458</td>
<td>1062.31</td>
</tr>
<tr>
<td>umd-Simulated</td>
<td>-0.0001</td>
<td>-0.0354</td>
<td>-116.3571</td>
<td>3800.55</td>
</tr>
<tr>
<td>(t)</td>
<td>(4.02)***</td>
<td>(6.71)***</td>
<td>(-1.46)</td>
<td>(1.73)*</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: Performance Measures and Utility Comparison</th>
<th>(S)</th>
<th>(P_{AS})</th>
<th>(P_{FH})</th>
<th>(u)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{umd})</td>
<td>0.1675</td>
<td>0.0630</td>
<td>0.0288</td>
<td>-0.2486</td>
</tr>
<tr>
<td>Simulated</td>
<td>0.1673</td>
<td>0.0772</td>
<td>0.0494</td>
<td>-0.2475</td>
</tr>
<tr>
<td>umd-Simulated</td>
<td>0.0001</td>
<td>-0.0142</td>
<td>-0.0206</td>
<td>-0.0012</td>
</tr>
<tr>
<td>(t)</td>
<td>(3.49)***</td>
<td>(3.15)***</td>
<td>(597.26)***</td>
<td>(-89.71)***</td>
</tr>
</tbody>
</table>

Further, Column (3) reports the difference between the true and the simulated distributions, showing the statistical significance of the difference with asterisks.
Table 5: Private vs. Public Equity

This table compares the performance of private (Column (1)) and public (Column (2)) equity. The private equity performance is evaluated based on entrepreneur-level returns constructed from the 2004 Survey of Consumer Finance (SCF). This sample is conditional on survival, yielding an upward bias in the evaluation of private equity performance. The public equity returns are proxied by returns to the CRSP value-weighted market index. The difference between private and public equity performance is reported in Column (3).

Panel A shows the GMM estimates of various moments of the equity returns, where $\mu_1$ denotes the first moment, and $m_2$, $m_3$, and $m_4$ stand for the second, third and fourth central moments, respectively. Panel B reports the Sharpe ratio ($S$) and the Aumann-Serrano and Foster-Hart performance measures ($P^{AS}$ and $P^{FH}$). Further, Panel B also reports the expected utility ($u$) of a risk-averse investor with CRRA utility and a risk aversion coefficient of 5, assuming that the initial wealth and the investment scale are equal. The t-statistics are reported in the parentheses below the corresponding estimates. Asterisks denote statistical significance at the 1% (***) , 5% (**) and 10% (*) levels.

### Panel A: Various Moment

<table>
<thead>
<tr>
<th></th>
<th>(1) Private</th>
<th>(2) Public</th>
<th>(3) Private-Public</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_1$</td>
<td>196.5432</td>
<td>5.5895</td>
<td>190.9537</td>
</tr>
<tr>
<td></td>
<td>(0.26)***</td>
<td>(5.11)***</td>
<td></td>
</tr>
<tr>
<td>$m_2$</td>
<td>$6.1506 \times 10^6$</td>
<td>$16.2216$</td>
<td>$6.1506 \times 10^6$</td>
</tr>
<tr>
<td></td>
<td>(1.90)*</td>
<td>(23.29)***</td>
<td>(1.90)*</td>
</tr>
<tr>
<td>$m_3$</td>
<td>$5.1344 \times 10^{11}$</td>
<td>$73.0088$</td>
<td>$5.1344 \times 10^{11}$</td>
</tr>
<tr>
<td></td>
<td>(1.75)*</td>
<td>(8.58)***</td>
<td>(1.75)*</td>
</tr>
<tr>
<td>$m_4$</td>
<td>$4.6177 \times 10^{16}$</td>
<td>$2400.33$</td>
<td>$4.6177 \times 10^{16}$</td>
</tr>
<tr>
<td></td>
<td>(1.74)*</td>
<td>(15.69)***</td>
<td>(1.74)*</td>
</tr>
</tbody>
</table>

### Panel B: Performance Measures and Utility Comparison

<table>
<thead>
<tr>
<th></th>
<th>(1) Private</th>
<th>(2) Public</th>
<th>(3) Private-Public</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td>0.0793</td>
<td>1.3878</td>
<td>-1.3086</td>
</tr>
<tr>
<td></td>
<td>(10.57)***</td>
<td>(50.26)***</td>
<td>(-45.96)***</td>
</tr>
<tr>
<td>$P^{AS}$</td>
<td>0.0549</td>
<td>0.7971</td>
<td>-0.7422</td>
</tr>
<tr>
<td></td>
<td>(38.55)***</td>
<td>(41.59)***</td>
<td>(-38.18)***</td>
</tr>
<tr>
<td>$P^{FH}$</td>
<td>0.0100</td>
<td>0.2583</td>
<td>-0.2483</td>
</tr>
<tr>
<td></td>
<td>(174466)***</td>
<td>(47694232)***</td>
<td>(-4293137)***</td>
</tr>
<tr>
<td>$u$</td>
<td>-25452.13</td>
<td>-0.2039</td>
<td>-25451.93</td>
</tr>
<tr>
<td></td>
<td>(-2.24)***</td>
<td>(-451.73)***</td>
<td>(-2.24)**</td>
</tr>
</tbody>
</table>
Table 6: Active vs. Passive Funds

This table compares the performance of actively managed (Column (1)) and index (Column (2)) funds, based on the CRSP mutual fund return data for the time period January 1991 to December 2009. The difference between active and passive fund performance is reported in Column (3). Panel A shows the GMM estimates of various moments of the fund returns, where \( \mu_1 \) denotes the first moment, and \( m_2, m_3, \) and \( m_4 \) stand for the second, third and fourth central moments, respectively. Panel B reports the Sharpe ratio \( (S) \) and the Aumann-Serrano and Foster-Hart performance measures \( (P^{AS} \) and \( P^{FH} \)). Further, Panel B also reports the expected utility \( (u) \) of a risk-averse investor with CRRA utility and a risk aversion coefficient of 5, assuming that the initial wealth and the investment scale are equal. The t-statistics are reported in the parentheses below the corresponding estimates. Asterisks denote statistical significance at the 1% (***) , 5% (**) and 10% (*) levels.

### Panel A: Various Moment

<table>
<thead>
<tr>
<th></th>
<th>(1) Active</th>
<th>(2) Passive</th>
<th>(3) Active-Passive</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_1 )</td>
<td>0.5210</td>
<td>0.5409</td>
<td>-0.0199</td>
</tr>
<tr>
<td></td>
<td>(1.79)*</td>
<td>(1.88)*</td>
<td>(-0.30)</td>
</tr>
<tr>
<td>( m_2 )</td>
<td>19.2097</td>
<td>18.9356</td>
<td>0.2741</td>
</tr>
<tr>
<td></td>
<td>(7.90)***</td>
<td>(8.58)***</td>
<td>(0.44)</td>
</tr>
<tr>
<td>( m_3 )</td>
<td>-67.3233</td>
<td>-58.9134</td>
<td>-8.4099</td>
</tr>
<tr>
<td></td>
<td>(-2.24)**</td>
<td>(-2.17)**</td>
<td>(-1.38)</td>
</tr>
<tr>
<td>( m_4 )</td>
<td>1718.30</td>
<td>1610.39</td>
<td>107.91</td>
</tr>
<tr>
<td></td>
<td>(2.81)***</td>
<td>(2.96)***</td>
<td>(0.97)</td>
</tr>
</tbody>
</table>

### Panel B: Performance Measures and Utility Comparison

<table>
<thead>
<tr>
<th></th>
<th>(1) Active</th>
<th>(2) Passive</th>
<th>(3) Active-Passive</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S )</td>
<td>0.1189</td>
<td>0.1243</td>
<td>-0.0054</td>
</tr>
<tr>
<td></td>
<td>(1.71)*</td>
<td>(1.79)*</td>
<td>(-0.37)</td>
</tr>
<tr>
<td>( P^{AS} )</td>
<td>0.0508</td>
<td>0.0537</td>
<td>-0.0029</td>
</tr>
<tr>
<td></td>
<td>(1.70)*</td>
<td>(1.78)*</td>
<td>(-0.40)</td>
</tr>
<tr>
<td>( P^{FH} )</td>
<td>0.0441</td>
<td>0.0465</td>
<td>-0.024</td>
</tr>
<tr>
<td></td>
<td>(2.26)**</td>
<td>(2.41)**</td>
<td>(-0.64)</td>
</tr>
<tr>
<td>( u )</td>
<td>-0.2500</td>
<td>-0.2497</td>
<td>-0.0003</td>
</tr>
<tr>
<td></td>
<td>(-75.41)***</td>
<td>(-76.96)***</td>
<td>(-0.49)</td>
</tr>
</tbody>
</table>
Table 7: Mutual Fund Selection Based on Performance Measures

This table reports the performance of investments following monthly rebalancing strategies based on the \(P^{AS}\) measure (Columns (1)), the \(P^{FH}\) measure (Column (2)) and the Sharpe ratio (Column (3)) for the period January 1967 to December 2009. For comparison, the table also reports the performance of investing in the market, represented by the CRSP value-weighted index (Column (4)) for the same sample period. Columns (5)-(10) show the difference in performance from investing in the various portfolios. Panel A reports the GMM estimates of various moments of the portfolio returns, where \(\mu_1\) denotes the first moment, and \(m_2\), \(m_3\), and \(m_4\) stand for the second, third and fourth central moments, respectively. Also reported in Panel A are the Sharpe ratio \((S)\) and the Aumann-Serrano and Foster-Hart performance measures \((P^{AS}\) and \(P^{FH}\)). The t-statistics are reported in the parentheses below the corresponding estimates. Asterisks denote statistical significance at the 1\% (***), 5\% (**) and 10\% (*) levels.

<table>
<thead>
<tr>
<th></th>
<th>(1) AS</th>
<th>(2) FH</th>
<th>(3) S</th>
<th>(4) MKT</th>
<th>(5) AS-FH</th>
<th>(6) AS-S</th>
<th>(7) AS-MKT</th>
<th>(8) FH-S</th>
<th>(9) FH-MKT</th>
<th>(10) S-MKT</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mu_1)</td>
<td>0.4961</td>
<td>0.4671</td>
<td>0.5383</td>
<td>0.4269</td>
<td>0.0290</td>
<td>-0.0122</td>
<td>0.0692</td>
<td>-0.0712</td>
<td>0.0402</td>
<td>0.1114</td>
</tr>
<tr>
<td>(m_2)</td>
<td>16.1544</td>
<td>15.6887</td>
<td>21.2108</td>
<td>21.4319</td>
<td>0.4657</td>
<td>-5.0564</td>
<td>-5.2774</td>
<td>-5.5221</td>
<td>-5.7432</td>
<td>-0.2211</td>
</tr>
<tr>
<td>(m_3)</td>
<td>-37.2520</td>
<td>-36.9873</td>
<td>-59.1041</td>
<td>-55.5719</td>
<td>-0.2647</td>
<td>21.8521</td>
<td>18.3199</td>
<td>22.1168</td>
<td>18.5846</td>
<td>-3.5221</td>
</tr>
<tr>
<td>(m_4)</td>
<td>1312.89</td>
<td>1284.98</td>
<td>2090.75</td>
<td>2264.29</td>
<td>27.91</td>
<td>77.86</td>
<td>80.57</td>
<td>79.31</td>
<td>73.55</td>
<td>-0.48</td>
</tr>
<tr>
<td>(S)</td>
<td>0.1234</td>
<td>0.1179</td>
<td>0.1169</td>
<td>0.0922</td>
<td>0.0055</td>
<td>0.0066</td>
<td>0.0312</td>
<td>0.0011</td>
<td>0.0257</td>
<td>0.0247</td>
</tr>
<tr>
<td>(P^{AS})</td>
<td>0.0582</td>
<td>0.0565</td>
<td>0.0482</td>
<td>0.0384</td>
<td>0.0018</td>
<td>0.0100</td>
<td>0.0199</td>
<td>0.0082</td>
<td>0.0181</td>
<td>0.0099</td>
</tr>
<tr>
<td>(P^{FH})</td>
<td>0.0497</td>
<td>0.0484</td>
<td>0.0424</td>
<td>0.0347</td>
<td>0.0013</td>
<td>0.0074</td>
<td>0.0150</td>
<td>0.0061</td>
<td>0.0137</td>
<td>0.0076</td>
</tr>
</tbody>
</table>
Panel B: Utility Comparison

This panel reports the expected utility (\( u \)) of a risk-averse investor with CRRA utility and risk aversion coefficients of 3, 5, 10, 15, and 20, which are typical in asset pricing calibration exercises. We assume that the initial wealth and the investment scale are equal. The t-statistics are reported in the parentheses below the corresponding estimates. Asterisks denote statistical significance at the 1% (***) , 5% (**) and 10% (*) levels.

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
<th>(7)</th>
<th>(8)</th>
<th>(9)</th>
<th>(10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>-0.4976</td>
<td>-0.4978</td>
<td>-0.4979</td>
<td>-0.4991</td>
<td>0.0002</td>
<td>0.0004</td>
<td>0.0015</td>
<td>0.0002</td>
<td>0.0013</td>
<td>0.0011</td>
</tr>
<tr>
<td></td>
<td>(-268.86)***</td>
<td>(-272.40)***</td>
<td>(-232.23)***</td>
<td>(-230.54)***</td>
<td>(1.40)</td>
<td>(0.82)</td>
<td>(2.42)**</td>
<td>(0.30)</td>
<td>(1.96)**</td>
<td>(1.77)**</td>
</tr>
<tr>
<td>5</td>
<td>-0.2493</td>
<td>-0.2495</td>
<td>-0.2503</td>
<td>-0.2515</td>
<td>0.0002</td>
<td>0.0099</td>
<td>0.0022</td>
<td>0.0008</td>
<td>0.0020</td>
<td>0.0012</td>
</tr>
<tr>
<td></td>
<td>(-127.67)***</td>
<td>(-129.14)***</td>
<td>(-109.21)***</td>
<td>(-107.62)***</td>
<td>(1.08)</td>
<td>(1.92)**</td>
<td>(3.11)***</td>
<td>(1.34)</td>
<td>(2.75)***</td>
<td>(1.68)**</td>
</tr>
<tr>
<td>10</td>
<td>-0.1155</td>
<td>-0.1155</td>
<td>-0.1184</td>
<td>-0.1199</td>
<td>0.00007</td>
<td>0.0029</td>
<td>0.0014</td>
<td>0.0029</td>
<td>0.0014</td>
<td>0.0015</td>
</tr>
<tr>
<td></td>
<td>(-46.71)***</td>
<td>(-46.86)***</td>
<td>(-38.60)***</td>
<td>(-35.46)***</td>
<td>(0.37)</td>
<td>(3.73)***</td>
<td>(3.40)***</td>
<td>(3.29)***</td>
<td>(3.40)***</td>
<td>(1.22)</td>
</tr>
<tr>
<td>15</td>
<td>-0.0824</td>
<td>-0.0824</td>
<td>-0.0886</td>
<td>-0.0914</td>
<td>-0.000004</td>
<td>0.0062</td>
<td>0.0090</td>
<td>0.0062</td>
<td>0.0090</td>
<td>0.0028</td>
</tr>
<tr>
<td></td>
<td>(-22.48)***</td>
<td>(-22.20)***</td>
<td>(-17.90)***</td>
<td>(-18.81)***</td>
<td>(-0.01)</td>
<td>(4.93)***</td>
<td>(2.49)***</td>
<td>(3.89)***</td>
<td>(2.59)***</td>
<td>(0.88)</td>
</tr>
<tr>
<td>20</td>
<td>-0.0736</td>
<td>-0.0736</td>
<td>-0.0860</td>
<td>-0.0939</td>
<td>-0.00002</td>
<td>0.0123</td>
<td>0.0203</td>
<td>0.0123</td>
<td>0.0203</td>
<td>0.0079</td>
</tr>
<tr>
<td></td>
<td>(-11.68)***</td>
<td>(-11.29)***</td>
<td>(-9.20)***</td>
<td>(-5.71)***</td>
<td>(-0.03)</td>
<td>(3.63)***</td>
<td>(1.80)</td>
<td>(3.64)***</td>
<td>(1.87)**</td>
<td>(0.80)</td>
</tr>
</tbody>
</table>
Figure 1: Distributions of Anomaly Returns
Figure 2: Distributions of Momentum and Simulated Normal Returns
Figure 3: Distributions of Private and Public Equity Returns (winsorized at 200%)
Figure 4: Distributions of Active and Passive Fund Returns