Information Acquisition in Rumor-Based Bank Runs

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Abstract

We study information acquisition and withdrawal decisions when a liquidity event triggers a spreading rumor and exposes a bank to a run. Uncertainty about the bank’s liquidity and potential failure motivates agents who hear the rumor to acquire additional signals. Depositors with unfavorable signals gradually run on the bank. A bank run equilibrium exists when agents aggressively acquire information. We study threshold parameters that eliminate runs. Public provision of solvency information can eliminate runs by indirectly crowding-out individual depositors’ effort to acquire liquidity information. However, providing too much information that slightly differentiates competing solvent-but-illiquid banks can result in inefficient runs.

JEL Classification: D8, G2

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1 Introduction

Bank runs are unfortunately still with us. Runs occurred recently on major traditional banks such as Northern Rock and IndyMac, and on non-traditional “shadow” banks.\(^1\) The largest ever bank failure resulted from the 2008 run on Washington Mutual (WaMu). Figure 1 shows that the run on WaMu was dynamic in nature: the bank experienced the gradual withdrawal of $16 billion in the days leading up to its takeover by the FDIC. Uncertainty regarding WaMu’s liquidity and hence its eventual fate remained heightened during the run.\(^2\) This empirical pattern that runs have an important time dimension, and that uncertainty and learning play a key role, has been found as far back as the 1857 run on the Emigrant Industrial Savings Bank.\(^3\)

In this paper, we provide the first dynamic bank run model featuring learning and endogenous information acquisition to shed light on two important questions: Will endogenous information acquisition exacerbate the inefficient bank run equilibrium? What are the policy implications for a social planner aiming to eliminate run equilibrium?

We incorporate uncertainty about the bank’s liquidity into the asynchronous awareness framework of Abreu and Brunnermeier (2003).\(^4\) At some unobservable random time, a liquidity event occurs for a fundamentally solvent bank, at which point it becomes illiquid (i.e., limited amount of cash reserves and susceptible to a run) or remains liquid (with sufficient reserves and not subject to a run). The liquidity event triggers the spread of a rumor in the population that exposes the bank to a run. Agents learn about whether the bank is liquid from the passage of time without failure, and about the time the liquidity event started.

Uncertainty motivates informed agents who hear the rumor to acquire additional information

\(^1\)Calomiris and Mason (1997), Shin (2009), and Iyer and Puri (forthcoming) document traditional bank runs, while Gorton and Metrick (2011) and Acharya, Schnabl, and Suarez (forthcoming) document “shadow” bank runs on repo and asset-backed commercial paper conduits.

\(^2\)On September 16, 2008, “’This week and next will be the moment of truth for WaMu,’ said Fred Cannon, an analyst at Keefe Bruyette & Woods.” See report “WaMu faces price to keep deposits,” Financial Times, by Saskia Scholtes in New York.

\(^3\)O Grada and White (2003) find that the 1857 bank run on the Emigrant Industrial Savings Bank had an important time dimension and was driven by informational shocks in the face of asymmetric information about the true condition of bank portfolios. See Kelly and O Grada (2000) for evidence on information transmission during the run.

\(^4\)Abreu and Brunnermeier (2002, 2003) consider the asynchronous timing of awareness to study bubbles. In their model, delayed arbitrage is rational because agents do not know whether other agents are aware of this arbitrage opportunity, and more importantly, when they are taking action on it. Brunnermeier and Morgan (2010) generalize this idea to a class of “clock games” and test its main predictions in controlled experiments. We add to their model uncertainty about the capacity of the bubble (bank), allow agents to acquire additional information upon awareness, and decouple the spreading rate from the length of the awareness window.
Daily net change in deposits as reported by Washington Mutual Bank to the Office of Thrift Supervision (OTS). We take out Friday and end-of-month fixed effects, i.e. days with automatic payday deposits estimated over the preceding 52 days. The OTS appointed the FDIC as receiver of WaMu on the evening of September 25, 2008 (the vertical line in the figure) which then sold it to JPMorgan Chase.

at an endogenously determined signal quality. The realized signal may be utterly uninformative, or reveals the bank liquidity state perfectly. The higher the signal’s quality, the more likely is the signal to reveal the bank’s liquidity status. If the bank is indeed illiquid, then in aggregate a higher quality translates to a greater fraction of agents who know the bank is illiquid.

The presence of agents with heterogeneous information naturally allows us to derive the unique bank run equilibrium as an interior solution, given the signal quality about bank illiquidity.\(^5\) Conditional on the bank being illiquid, agents with informative “low” signals perceive a high bank failure hazard rate and withdraw immediately, while agents with uninformative “medium” signals wait. When all agents with uninformative medium signals wait a bit longer, they know more agents with informative low signals have already withdrawn. As a result, the bank failure hazard rate endogenously rises because more agents run earlier. This “fear-of-low-signal-agents” generates a time-varying marginal cost of waiting for agents with the medium signal, which allows us to find the unique endogenous withdrawal time that equates the marginal benefit of waiting. Waiting is an interesting new dimension of runs that cannot be analyzed in static models.\(^6\)

\(^5\)In Abreu and Brunnermeier (2003), the interior equilibrium is generated under the assumption that the bubble component is exogenously decreasing over time. This time-varying attacking benefit drives arbitragers to attack only when the bubble component drops to certain level.

\(^6\)Our model belongs to a vast literature on the role of information in static bank runs, and it is beyond the scope of our paper to have a thorough review on this topic. To name a few, Gorton (1985), Bhattacharya and Gale (1987),
We decouple the speed by which information spreads from the length of the awareness window over which information spreads and examine their effects on the survival of an illiquid bank.\footnote{By contrast, in Abreu and Brunnermeier (2003) the speed by which information spreads is one over the length of the awareness window.} In a static model an increase in the number of potential withdrawers intuitively exacerbates runs. By contrast, we find that when the awareness window widens so everybody knows potentially more agents run on the bank, the illiquid bank survives longer. This counter-intuitive result arises due to the novel uncertainty structure we introduce, as the agent who hears the rumor also observes that the bank is still alive. If the awareness window is wide, the rumor could have started spreading a long time before he heard it. Thus, conditional on the bank surviving thus far, the bank is more likely to be liquid, leading to lower running incentives.

Given the unique bank run equilibrium, the optimal signal quality for individual agents trades off the convex cost of acquiring it with a greater probability of receiving an informative signal. Moreover, the chosen quality of information affects the equilibrium survival time of the failing bank. This intricate feedback effect between bank runs and information acquisition generally leads to at most two equilibrium: a good equilibrium where agents do not acquire information and do not run, and a bad equilibrium where agents acquire information aggressively and run on the illiquid bank. This result naturally leads us to examine how to eliminate the run equilibrium.

We find that in our dynamic setting the threshold parameters that eliminate bank runs altogether are nontrivial. The social planner can, for example, adjust bank reserve requirements in response to heterogeneity in the rumor spreading rate and in the length of the awareness window. Interestingly, the liquidity requirement is far less than the maximum potential withdrawal, which is the minimum reserve requirement from a static Diamond and Dybvig perspective. In fact, we can rule out pure-strategy synchronized sun-spot equilibrium. Moreover, partial deposit insurance suffices in our setting to prevent runs. Though beyond the scope of the current paper, a calibration of our model to data can certainly result in quantitatively meaningful policy recommendations.

We extend our baseline model along two dimensions. First, we consider fundamentally insolvent...
banks. When agents privately collect information about bank solvency, they inevitably learn about bank liquidity. Effort spent acquiring one reduces the effort needed to acquire the other. We show that public provision of solvency information can help curb the private information acquisition effort on bank liquidity. One such situation where information is key is at the disclosure of stress test results (Bernanke, 2010). Carefully constructed stress tests can therefore help prevent bank runs by crowding out information acquisition by individuals.

The planner must be careful, however, to avoid providing too much information that differentiates competing solvent but potentially illiquid banks, for such information can start a run. Competition is the subject of our second extension. In a competitive environment, small differences in liquidity can result in runs on slightly weaker banks and their subsequent failure. In this case, the planner would want to inject noise into the system, so that individual agents with less informative signals are more likely to stay in their original bank without knowing which one is more liquid. For example, by forcing all of the “Big 9” banks to take government capital on October 13, 2008, the U.S. government was in fact injecting noise about the liquidity of competing solvent banks into the economy.

Our paper contributes to a recent theoretical literature studying dynamic bank runs. Wallace (1988) and Green and Lin (2003) are earlier papers studying the sequential service constraint in the Diamond and Dybvig model. Gu (2011) is a dynamic model in nature, that studies depositors’ withdrawal strategy sequentially, and He and Xiong (forthcoming) develop a continuous-time debt run model of a firm with a time-varying fundamental and a staggered debt structure. Relative to these models, we emphasize the role of information and learning play in runs, and how the government can eliminate runs when individuals acquire information.

The paper proceeds as follows. Section 2 describes the setting and solves the agent’s learning problem. Section 3 characterizes the individual agent’s optimal withdrawal policy, and Section 4 analyzes the bank run equilibrium with information acquisition. Section 5 considers extensions, and we conclude in Section 6. Proofs are in the Appendix.

2 The Model

We describe the economy and the individual agent’s problem, with the focus on the agent’s learning.
2.1 The Setting

2.1.1 Technology

Time is continuous on $t \in [0, \infty)$. A continuum of infinitely-lived risk-neutral agents (depositors) with unit mass maximize their expected utility from consumption with a zero discount rate.

Bank deposits yield a constant rate of return $r > 0$ when the bank is operating, while holding cash outside the bank earns zero return. Broadly, one can interpret the bank as some investment vehicle in the shadow banking system or even the entire financial system, and the positive relative wedge $r > 0$ reflects either a higher investment growth rate or a convenience yield for keeping funds in the institution. To avoid exploding values, we assume that the bank’s growth stops at some “maturing” event modeled as a Poisson shock with intensity $\delta > r$. Following this public event, the growth-less bank liquidates without loss, and the game ends in the sense that each agent gets his deposit back for consumption.\(^8\) Throughout, this maturing event will be independent of any other random variables that we consider.

2.1.2 Uncertainty about Bank Liquidity

There are two potential types of banks that are fundamentally solvent, with one type of bank being “illiquid,” and a second “liquid” bank impervious to runs. The uncertainty is crucial to our analysis. Later we will introduce insolvent banks as an extension.

Throughout, bank liquidity is defined as the amount of depositors that it takes to run down the bank. For simplicity, we assume a binary structure for the state of bank liquidity $\tilde{\kappa}$. For the liquid bank, $\tilde{\kappa} = \kappa_H > 1$, i.e., the bank can survive any severe run.\(^9\) However, when the bank is illiquid, $\tilde{\kappa}$ takes a lower value $\kappa_L < 1$, and the bank is potentially subject to runs. In other words, the illiquid bank fails when more than a $\kappa$ measure of the depositors have fully withdrawn their funds. One can literally interpret $\tilde{\kappa}$ as the bank’s cash reserves to meet withdrawals. We broadly interpret $\tilde{\kappa}$ as the liquidity of the bank.

In our economy, upon awareness (to be modeled shortly), informed agents begin with a prior\(^8\)\(^9\)

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\(^8\)The maturing event assumption plays no role in our analysis except making the value of a dollar inside the bank finite (as the liquid bank will grow always). Alternatively, we could assume that each individual agent suffers liquidity shocks that require immediate consumption (therefore withdrawal). The analysis, which is available upon request, is similar but much more complicated (due to the replacement of agents).

\(^9\)There is not much loss in generality from assuming $\kappa_H > 1$ since we can always pick an interior $\kappa_H < 1$ such that runs do not occur in equilibrium. See Section 5 for the analysis of minimum liquidity $\kappa$ to eliminate runs.
that \( \Pr \{ \bar{\kappa} = \kappa_L \} = p_0 \in (0, 1) \). When the illiquid bank fails, all remaining depositors in the failed bank recover a fraction \( \gamma \in (0, 1) \) of their deposits. Afterwards, all agents go to autarky consuming their remaining wealth.

2.1.3 Liquidity Event and Spreading Rumors

At \( t = 0 \) the bank is liquid, i.e., \( \bar{\kappa} = \kappa_H \). At some unobservable random time \( \bar{t}_0 > 0 \) which we refer to as the liquidity event, the bank may become illiquid, and the uncertainty is as modeled in Section 2.1.2.

Moreover, this liquidity event triggers information to spread in the population. We call those agents who have heard the information as “informed,” and those who have not as “uninformed.” Since at \( t = 0 \) the bank is liquid, we assume that the agents’ beliefs are such that they expect to hold money in the bank unless they hear the information that the bank might be illiquid.

Although the exact liquidity event time \( t_0 \) is publicly unobservable, it is common knowledge that \( \bar{t}_0 \) is exponentially distributed on \([0, \infty)\) with cumulative distribution function \( \Phi (t_0) \equiv 1 - e^{-\theta t_0} \), and density \( \phi(t_0) \equiv \theta e^{-\theta t_0} \). This information structure is meant to capture the essence of an unverified rumor of uncertain origin that spreads gradually in the depositor population. We call this information about bank liquidity a “rumor.”

Given a realization of \( \bar{t}_0 = t_0 \), the rumor begins to spread over an interval \([t_0, t_0 + \eta]\) with a positive constant (exogenous) length \( \eta \). Following Abreu and Brunnermeier (2003) we refer to \( \eta \) as the “awareness window.” At any interval \( dt \) where \( t \in (t_0, t_0 + \eta) \), uninformed agents become informed by receiving this signal with probability \( \beta dt \). This information shock is i.i.d. across the population of uninformed agents. To make the problem interesting, at time \( t_0 + \eta \) the fraction of informed agents is sufficient to take down the bank, if they decide to do so.

This information spreading technology is different from that of Abreu and Brunnermeier (2003) in that we allow for separation between the spreading rate \( \beta \) and the awareness window \( \eta \). Abreu and Brunnermeier assume a linear spreading technology with rate \( \frac{1}{\eta} \) so that the entire population is informed at \( t_0 + \eta \). This way, the awareness window \( \eta \) is artificially tied to the spreading rate \( \frac{1}{\eta} \). As we show later, because of the endogenous learning, the awareness window \( \eta \) has interesting effects opposite to common wisdom.

Given \( t_0 \) and some future time \( t \in [t_0, t_0 + \eta] \), denote by \( U (t|t_0) \in [0, 1] \) the measure of unin-
formed agents who have not heard the rumor yet. At $t = t_0$, every agent is uninformed, so that $U (t_0|t_0) = 1$. The population of uninformed agents evolves as $dU (t|t_0) = -\beta U (t|t_0) dt$, which implies that $U (t|t_0) = e^{-\beta (t-t_0)}$. Hence, the mass of newly informed agents within $[t, t+dt]$ is

$$
\beta U (t|t_0) dt = \beta e^{-\beta (t-t_0)} dt \text{ for } t \in [t_0, t_0 + \eta].
$$

(1)

2.1.4 Information Acquisition

At time $t_i$ when the agent hears the rumor, he can acquire additional information about the bank’s type at some convex cost. Specifically, the agent makes an endogenous choice of information quality $q \in [0, 1]$ with cost $\chi (q) = \alpha q^2/2$, where $\alpha > 0$ is a positive constant. For tractability, we assume $\chi (q)$ is the per dollar information cost, so informed agents face the same problem when they are informed at different times.

This additional signal takes three possible values $y \in \{y_L, y_M, y_H\}$ with conditional probabilities:

$$
\Pr \{y = y_H|\kappa = \kappa_H\} = q, \Pr \{y = y_M|\kappa = \kappa_H\} = 1 - q, \text{ and } \Pr \{y = y_L|\kappa = \kappa_H\} = 0;
$$

$$
\Pr \{y = y_L|\kappa = \kappa_L\} = q, \Pr \{y = y_M|\kappa = \kappa_L\} = 1 - q, \text{ and } \Pr \{y = y_H|\kappa = \kappa_L\} = 0. \quad (2)
$$

Figure 2 summarizes this distribution. With probability $q$, the bank’s liquidity is perfectly revealed by the signal $y_H (y_L)$. With probability $1 - q$, the agent does not learn anything by receiving the
medium signal \(y_M\).\(^{10}\) The realizations of these signals are i.i.d. across agents, conditional on the underlying state.

### 2.1.5 Information Structure

Call the agent who is informed at \(t_i \geq t_0\) simply agent \(t_i\). Denote by \(\mathcal{F}_{t_i}^t\) his information set at \(t > t_i\), and \(1_{t_i}^{BF} \in \{1, 0\}\) the indicator of whether the bank failed or not by time \(t\). Agent \(t_i\)'s information set is \(\mathcal{F}_{t_i}^t = \{t, t_i, \bar{y}_{t_i}, 1_{t_i}^{BF}\}\), i.e. the current time, the time he hears the rumor, the additional signal he acquired, and whether the bank has failed or not. Recall that the liquidity event \(t_0\) is not public information.

### 2.1.6 Agent’s Problem

Now we summarize the problem of an informed agent \(t_i\) with deposits inside the bank. After the acquisition of an additional signal \(\bar{y} \in \{y_L, y_M, y_H\}\), the agent can withdraw his deposits whenever he believes bank failure is imminent, and redeposit this cash in the future if the bank’s survival sufficiently improves his posterior belief about bank liquidity. To eliminate strategies with infinite transactions, we assume a constant transaction cost \(k\) per dollar of deposits when the agent (re)deposits his cash into the bank.

Risk neutrality and the bank’s superior investment technology imply it is optimal for the agent to consume only at the bank’s exogenous maturing event, or when the bank endogenously failed due to runs. Also, the linearity of this problem implies a “bang-bang” strategy, i.e. keeping the entire wealth either in or out of the bank, is optimal.

### 2.2 Learning

#### 2.2.1 Posterior Belief about \(t_0\)

Agent \(t_i\) updates his posterior distribution of \(t_0\) conditional on hearing the rumor at \(t_i\). Given \(t_0\), for an individual uninformed agent the probability of getting informed over \([t_i, t_i + dt]\) is determined as follows. For \(t_i \in [t_0, t_0 + \eta]\), the probability of becoming informed at \([t_i, t_i + dt]\) but not before \(t_i\) is

\[
 f(t_i|t_0) \equiv \beta U(t_i|t_0) dt = \beta e^{\beta (t_i - t_0)} dt.
\]

And, since the rumor stops spreading after \(t_0 + \eta\),

\[
\frac{\mu(1-q) + \beta e^{\beta (t_0 + \eta)} \mu}{\beta e^{\beta (t_0 + \eta)} (1-q) + \beta e^{\beta (t_0 + \eta)} \mu} = p_0.
\]

\(^{10}\)Upon receiving \(y_M\), the posterior probability of the bank being illiquid remains

\[
\frac{\mu(1-q) + \beta e^{\beta (t_0 + \eta)} \mu}{\beta e^{\beta (t_0 + \eta)} (1-q) + \beta e^{\beta (t_0 + \eta)} \mu} = p_0.
\]
we have \( f(t_i|t_0) = 0 \) for \( t_i \geq t_0 + \eta \). Combining with the density of \( \phi(t_0) = \theta e^{-\theta t_0} \), the informed agent \( t_i \) learns about the liquidity event timing \( t_0 \). Similar to Abreu and Brunnermeier (2003) we focus on realizations of \( t_0 \geq \eta \) such that the economy is already in this stationary phase; as shown shortly agent \( t_i \)'s equilibrium strategy will be independent of the exact timing of being informed.\(^\text{11}\)

With \( t_0 \geq \eta \), the informed agent \( t_i \geq t_0 \geq \eta \) updates his posterior belief about \( t_0 \) as

\[
\phi(t_0|t_i) \equiv \frac{f(t_i|t_0) \phi(t_0)}{\int_{t_0-\eta}^{t_i} f(t_i|s) \phi(s) ds} = \frac{\beta e^{-\beta(t_i-t_0)} \theta e^{-\theta t_0}}{\int_{t_0-\eta}^{t_i} \beta e^{-\beta(t_i-s)} \theta e^{-\theta s} ds} = \frac{\theta - \beta}{e^{(\theta-\beta)\eta} - 1} e^{(\theta-\beta)(t_i-t_0)}.
\]

Define \( \lambda \equiv \theta - \beta > 0 \) so the liquidity event intensity is greater than the rumor spreading rate.\(^\text{12}\)

Then

\[
\phi(t_0|t_i) = \frac{\lambda e^{\lambda(t_i-t_0)}}{e^{\lambda \eta} - 1}.
\]

Integrating (3) over \( t_0 \) we get the conditional distribution function for \( t_0 \):

\[
\Phi(t|t_i) \equiv \Pr(\bar{t}_0 \leq t|t_i) = \int_{t_0-\eta}^{t} \phi(s|t_i) ds = \begin{cases} 0 & t < t_i - \eta \\ \frac{e^{\lambda \eta} - e^{\lambda(t_i-\eta)}}{e^{\lambda \eta} - 1} & t_i - \eta \leq t \leq t_i \\ 1 & t > t_i \end{cases}
\]

2.2.2 Bank Failure Hazard Rate

Suppose that in a symmetric equilibrium, agents believe the illiquid bank fails at \( \bar{t}_0 + \zeta \) where \( \zeta \) is a constant to be determined in equilibrium (potentially infinite). Let \( F(t_0) \equiv t_0 + \zeta \) denote the failure time for a given realization of \( t_0 \), conditional on the bank being illiquid \( \kappa = \kappa_L \). The event of bank failure is \( \{F(t_0) < t, \kappa_L\} \), and if the bank fails at \( t \) then the inferred \( t_0 \) is

\[
t_0 = F^{-1}(t) = t - \zeta.
\]

Denote by \( p(t|t_i) \equiv \Pr(\bar{\kappa} = \kappa_L|\mathcal{F}_t^{t_i}) \) the posterior probability at time \( t \) of the bank being illiquid. Trivially, bank failure at \( t \) reveals that \( p(t|t_i) = 1 \). Also given \( y_L \) or \( y_H \) signals, \( p(t|t_i) = 1 \)

\(^{11}\)The finite awareness window \( \eta \) over which the rumor spreads makes the cases of \( t_0 < \eta \) and \( t_0 \geq \eta \) different. In the event of \( t_0 \geq \eta \), it always holds that \( t_i \geq \eta \), and rational agents know that \( t_0 \in [t_i - \eta, t_i] \). In Appendix B we consider the equilibrium behavior for \( t_0 < \eta \).

\(^{12}\)The assumption of \( \lambda > 0 \) is for exposition purpose, and the analysis goes though if \( \lambda < 0 \). To see this, if \( \lambda < 0 \), then the conditional density \( \phi(t_0|t_i) \) has to be written as \( \frac{(-\lambda)e^{\lambda(t_i-t_0)}}{1-e^{\lambda \eta}} \) which is still positive.
or 0. For an agent $t_i$ with $y_M$ signal, if the bank has not failed at $t$ (i.e. $t < F(t_0)$), then his posterior belief of the bank being illiquid is (recall that $\kappa$ is independent of $t_0$):

$$\begin{align*}
p(t|t_i) &= \Pr\{\kappa_L|t < F(t_0), t_i\} = \frac{\Pr\{t < F(t_0)|\kappa_L, t_i\}\Pr\{\kappa_L\}}{\Pr\{t < F(t_0)|\kappa_L, t_i\}\Pr(\kappa_L) + \Pr(\kappa_H)} \\
&= \frac{[1 - \Phi(F^{-1}(t)|t_i)]p_0}{[1 - \Phi(F^{-1}(t)|t_i)]p_0 + 1 - p_0}, \tag{6}
\end{align*}$$

We can derive $p(t|t_i)$ in closed form using (4) and (5). Moreover, as shown later, in equilibrium if $\zeta$ is finite then $\zeta \leq \eta$ must hold. In this case, when agent $t_i$ hears the rumor, his posterior probability of the bank being illiquid is

$$p(t_i|t_i) = \frac{e^{\lambda\zeta} - 1}{(1 - p_0)(e^{\lambda\eta} - 1) + (e^{\lambda\zeta} - 1)p_0}.$$ 

The cumulative distribution of bank failure times $t_i + \tau$, conditional on hearing the rumor at $t_i$ and the bank has not failed by $t_i$, can be derived as

$$\Pi(t_i + \tau|t_i) \equiv p(t_i|t_i)\Pr\{t_i < F(t_0) \leq t_i + \tau|t_i, \kappa_L\} = p(t_i|t_i)\frac{\Phi(F^{-1}(t_i + \tau)|t_i) - \Phi(F^{-1}(t_i)|t_i)}{1 - \Phi(F^{-1}(t_i)|t_i)}.$$ 

The bank failure hazard rate from the perspective of the informed agent $t_i$ is

$$h(t_i + \tau|t_i) \equiv \frac{d\Pi(t_i + \tau|t_i)}{d\tau}. \tag{7}$$ 

We present the closed form expression for $h(t_i + \tau|t_i)$ in the following proposition.

**Proposition 1.** Suppose that the illiquid bank fails at $t_0 + \zeta$ where $\zeta \leq \eta$ which holds in equilibrium. Then the bank failure hazard rate is

$$h(\tau) \equiv h(t_i + \tau|t_i) = \frac{\lambda e^{\lambda(\zeta-\tau)p_0}}{(1 - p_0)(e^{\lambda\eta} - 1) + (e^{\lambda(\zeta-\tau)} - 1)p_0} \quad \text{for } \tau \in [0, \zeta]. \tag{8}$$

For $\tau > \zeta$, $h(\tau) = 0$ as the bank is revealed to be liquid.

We have three noteworthy observations. First, the hazard rate in (8) is independent of the absolute timing of agent $t_i$ becoming informed, therefore we can denote $h(t_i + \tau|t_i)$ by $h(\tau)$. This property guarantees the stationarity of our model.
Second, the hazard rate only depends on the remaining survival time from potential bank failure, i.e., $\zeta - \tau$. This is intuitive: because agents in our economy are uncertain about the exact timing of the liquidity event $t_0$ that started the rumor, for each agent the bank failure hazard rate will depend on how far the economy is away from the maximum potential failure time $t_0 + \zeta$. The equilibrium remaining survival time is important for our later analysis.

Third, the above analysis is carried out for $y_M$ agents who remain uncertain about the bank’s type. The analysis for $y_L$ ($y_H$) agents are straightforward by simply replacing $p_0$ with 1 (0) in equation (8).

With this learning process in mind, we define a bank run equilibrium as follows

**Definition 1.** A bank run equilibrium is a stationary Perfect Bayesian Nash equilibrium in which informed agents’ strategies depend on the time since they heard the rumor, and the bank survival time is finite.

Stationarity here follows from the hazard rate’s independence of the absolute timing of agent $t_i$ becoming informed. The agent’s strategy depends on $\tau = t - t_i$, but not $t$. Finally, a bank run means the illiquid bank fails at a finite $t_0 + \zeta < \infty$.

### 2.3 Parameter Restrictions

We impose the following parametrization conditions throughout the paper:

\[
\frac{p_0}{1-p_0} > e^{\lambda \eta} - 1. \quad (9)
\]

This condition implies the bank hazard rate given in Proposition 1 is increasing with $\tau$, i.e. the time elapsed since hearing the rumor. Equivalently, this implies the bank failure hazard rate is decreasing in the remaining survival time $\zeta - \tau$.

Second, since the upper bound of the measure of informed agent is $1 - U(t_0 + \eta | t_0) = 1 - e^{-\beta \eta}$, for the model to be interesting we require that the illiquid bank can potentially fail if all informed agents run immediately, i.e.,

\[
1 - e^{-\beta \eta} > \kappa_L. \quad (10)
\]

Third, we focus on the case where both $y_M$ agents and $y_L$ agents are driving the bank failure.
Conditional on the bank being illiquid, there will be $q$ measure of $y_L$ agents. Therefore, instead of $q \in [0, 1]$, we further impose that the signal quality has to satisfy

$$q \in \left[0, \frac{\kappa_L}{1 - e^{-\beta \eta}}\right].$$

(11)

Under this condition, because signal quality $q < \frac{\kappa_L}{1 - e^{-\beta \eta}}$, agents with $y_L$ signal alone are not enough to take down the bank.

Fourth, we require that the maturity shock intensity $\delta$ is moderate:

$$\frac{\delta (1 - p_0) \left(e^{\lambda \eta} - 1\right) \frac{r(\lambda - k \delta)}{\delta - \lambda}}{\lambda (r - \lambda (1 - \gamma)) p_0} \in (0, 1).$$

(12)

This requirement implies $r - \lambda (1 - \gamma) > 0$. The only purpose of this condition is to guarantee the optimality of thresholds strategy.

Finally, we assume that the bank failure loss $1 - \gamma$ is significant:

$$(\lambda (1 - \gamma) - r) e^{\lambda \eta} + r > 0;$$

(13)

as we show in Section 3.3, this implies agents with a $y_L$ signal withdraw immediately, if the bank run equilibrium exists.

## 3 Optimal Withdrawal Strategies

In this section, we present the key proposition that characterizes the individual agent’s optimal withdrawal policy, taking both the equilibrium bank survival time $\zeta$ and information quality $q$ as given.

### 3.1 Value Functions

Denote by $V_I (\tau; t_i)$ ($V_O (\tau; t_i)$) the agent’s value of one dollar inside (outside) the bank at time $t_i + \tau$, where $\tau \geq 0$ is the time elapsed since agent $t_i$ heard the rumor. Due to stationarity, $V_I (\tau; t_i) = V_I (\tau)$ and $V_O (\tau; t_i) = V_O (\tau)$. Because withdrawal involves no transaction cost while redepositing costs $k$, $V_I (\tau) \geq V_O (\tau)$ and $V_O (\tau) \geq (1 - k) V_I (\tau)$ for all $\tau \geq 0$. 
When $\tau \geq \zeta$, the surviving bank is safe for sure. One dollar inside the bank will grow at $r$ until the maturing event (occurs with Poisson intensity $\delta$), which has a value of

$$V_I(\tau) = \int_0^\infty e^{rs} \delta e^{-\delta s} ds = \frac{\delta}{\delta - r} \text{ for } \tau \geq \zeta.$$  

The value of one dollar outside the bank is $V_O(\zeta) = (1 - k) V_I(\zeta) = \frac{(1-k)\delta}{\delta - r}$, which is above 1 for sufficiently small $k$.

When $\tau < \zeta$, consider a dollar outside the bank. At any point in time, if the status-quo position (i.e. keeping the dollar outside the bank) is optimal, then the following Hamilton-Jacobi-Bellman (HJB) equation must hold:

$$0 = h(\tau) (1 - V_O(\tau)) + \delta (1 - V_O(\tau)) + V'_O(\tau), \quad \text{Bank failure}$$

$$\text{Bank matures}$$

$$\text{Time change}$$

Here, the first term is the impact of bank failure: with hazard rate $h(\tau)$ the bank fails, and the agent loses his value $V_O(\tau)$ but recovers 1, the full value for his one dollar.\textsuperscript{13} The second term captures the bank asset maturing event, and the third term is the change due to time elapsing. Combined with the option of redepositing immediately (with transaction cost $k$), the HJB equation for one-dollar outside the bank is

$$0 = max \{ h(\tau) (1 - V_O(\tau)) + \delta (1 - V_O(\tau)) + V'_O(\tau), (1 - k) V_I(\tau) - V_O(\tau) \}$$

Similarly, for a dollar inside the bank, its value $V_I(\tau)$ must satisfy the following HJB equation:

$$0 = max \{ rV_I(\tau) + h(\tau) (\gamma - V_I(\tau)) + \delta (1 - V_I(\tau)) + V'_I(\tau), V_O(\tau) - V_I(\tau) \}$$

\textbf{3.2 Optimal Strategies}

Denote by $\hat{V}_O(\tau)$ the solution to the ordinary differential equation (14) with boundary condition $\hat{V}_O(\zeta) = V_O(\zeta)$, which is

$$\hat{V}_O(\tau) = \frac{e^{\lambda \eta} (1 - p_0) - 1 + e^{\lambda (\zeta - \tau)} p_0 + e^{-\delta (\zeta - \tau)} (1 - p_0) \left(e^{\lambda \eta} - 1\right) \frac{r - k \delta}{\delta - r}}{(1 - p_0) (e^{\lambda \eta} - 1) + (e^{\lambda (\zeta - \tau)} - 1) p_0}.$$  

\textsuperscript{13}The analysis holds for all signals as Proposition 1 gives $h(\tau)$ as a function of $p_0$. 

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In general \( V_O (\tau) \geq \tilde{V}_O (\tau) \): \( \tilde{V}_O (\tau) \) is the value at time \( \tau \) by simply staying outside the bank until \( \zeta \) (and redepositing at \( \zeta^+ \) if the bank is liquid and survives the run), and this simple continuation strategy may not be optimal. However, the following function captures the first-order impact of the withdrawal decision:

\[
g (\tau) \equiv h (\tau) (1 - \gamma) - r \tilde{V}_O (\tau). \tag{17}
\]

Here, \( g (\tau) \) is difference between the instantaneous loss due to potential bank failure (i.e. \( h (\tau) (1 - \gamma) \)), and \( r \tilde{V}_O (\tau) \) which is the instantaneous return of taking one dollar out now, keeping outside the bank until \( \zeta \), and redepositing back given survival. The underlying assumption here is that a threshold strategy is optimal (i.e. redepositing is never optimal before \( \zeta \)) which is verified in Proposition 2.

At the optimal withdrawal time \( \tau_w \), we have the first-order condition \( g (\tau_w) = 0 \), i.e.,

\[
h (\tau_w) (1 - \gamma) = r \tilde{V}_O (\tau_w) = r \left[ 1 + (1 - p (t_i + \tau_w | t_i)) e^{-\delta (\zeta - \tau_w) \frac{r - k \delta}{\delta - r}} \right], \tag{18}
\]

where we have used (16) and (6) in rewriting this intuitive expression. As mentioned above, the left hand side captures the marginal cost of staying, which is the hazard rate multiplying the bank failure loss. The right hand side captures the marginal benefit of staying, which is the growth rate \( r \) multiplying the agent’s continuation value of one dollar by withdrawing and redepositing after \( \zeta \). For the total continuation value in the bracket of right hand side in (18), the first term 1 is the principal amount which is present in Abreu and Brunnermeier (2003). The second term consists of the option value of future redepositing. Here, \( 1 - p (t_i + \tau_w | t_i) \) is the probability of the bank being liquid (and surviving eventually) conditional on bank survival at \( \tau_w \), \( e^{-\delta (\zeta - \tau_w)} \) is the discounting, and finally

\[
\frac{r - k \delta}{\delta - r} = V_O (\zeta) - 1 = \frac{(1 - k) \delta}{\delta - r} - 1 > 0
\]

is the additional payoff from redepositing. Intuitively, when \( p_0 = 1 \) so there is no uncertainty about bank liquidity, this option value term vanishes.

The next proposition shows formally that a threshold strategy is optimal based on the function \( g (\tau) \).

**Proposition 2.** *Given the equilibrium bank survival time \( \zeta \), the optimal policy for the agent with*
$y_M$ is as follows:

1. If $g(\zeta) \leq 0$, then it is optimal to stay in the bank always.
2. If $g(0) \geq 0$, then it is optimal to withdraw at 0 and redeposit right after $\zeta$.
3. Otherwise, we must have $g(0) < 0$ and $g(\zeta) > 0$, and there exists a unique waiting time $\tau_w \in (0, \zeta)$ so that $g(\tau_w) = 0$, and withdrawing at $\tau_w$ and redepositing right after $\zeta$ is optimal.

### 3.3 Values Conditional on Signals

We have studied the optimal strategy for the agent with $y_M$ signal. The agents with high signal $y_H$ should keep their deposits in the bank always. And, for agents who receive low signals, if

\[ g(0; p_0 = 1) = \frac{(\lambda(1 - \gamma) - r)e^{\lambda \zeta} + r}{e^{\lambda \zeta} - 1} > 0 \]

then it is optimal to withdraw immediately. Because $\zeta < \eta$ and $\lambda(1 - \gamma) < r$, (19) is implied by parameter condition (13). In fact, we will show later in Section 4.2 that condition (19) has to hold for any interior bank run equilibrium. In our model, if $y_L$ agents want to wait some positive amount of time, then generically bank run equilibrium do not exist.

**Proposition 3.** Suppose $g(0) < 0$ and $g(\zeta) > 0$ so that agents with $y_M$ signals wait $\tau_w$ given in (18). Then upon hearing the rumor, the values conditional on signals are

\[
V_I(0|y_L) = 1, V_I(0|y_H) = \frac{\delta}{\delta - r}, \\
V_I(0|y_M) = \frac{\delta (e^{\lambda \eta} - p_0 - 1)}{\delta - r} \left( 1 - e^{-(\delta - \gamma)\tau_w} \right) + \frac{\delta + \lambda \eta}{\lambda + \delta - r} e^{\lambda \zeta} p_0 + e^{-(\delta - \gamma)\tau_w} e^{\lambda (\zeta - \tau_w)} p_0 \left( \frac{\lambda + \delta (\lambda(1 - \gamma) - r)}{\lambda + \delta - r} \right)
\]

where $\tau_w$ satisfies the first-order condition in (18).

### 4 Bank Run Equilibrium

#### 4.1 Cumulative Withdrawals Given $q$

In our model, the illiquid bank fails at $t_0 + \zeta$ when aggregate cumulative withdrawals by informed agents deplete the illiquid bank’s capacity $\kappa_L$. Two groups of informed agents withdraw from the
Cumulative withdrawal patterns for an illiquid bank with capacity $\kappa_L$ and a liquid bank with capacity $\kappa_H > 1$. At $\tau = 0$ the rumor starts to spread. Agents with a $y_M$ signal wait in equilibrium $\tau_w = 0.85$ before withdrawing. The illiquid bank fails while the liquid bank starts experiencing redeposits at $\zeta = 1.8$. Parameter values are $r = 0.09$, $\beta = 1$, $\theta = 1.03$, $\eta = 2$, $p_0 = 0.8$, $\delta = 0.12$, $k = 10^{-6}$, $\kappa_L = 0.65$, $\alpha = 0.7$, $\gamma = 0.75$.

illiquid bank. The first group is $y_L$ agents with $q$ measure in aggregate per unit of time. Therefore, using (1), at $t_0 + \zeta$ total withdrawals by $y_L$ agents are

$$ q \int_{t_0}^{t_0+\zeta} \beta U(t_i|t_0) dt_i = q \left( 1 - e^{-\beta \zeta} \right). \tag{20} $$

Agents with $y_M$ signals wait for $\tau_w$, and at $t_0 + \zeta$ their total withdrawals are

$$ (1 - q) \int_{t_0}^{t_0+\zeta-\tau_w} \beta U(t_i|t_0) dt_i = (1 - q) \left( 1 - e^{-\beta(\zeta-\tau_w)} \right). \tag{21} $$

Figure 3 depicts the cumulative withdrawal patterns for both banks. For illiquid banks, $y_L$ agents begin to withdraw right after the liquidity event $t_0$ as the rumor starts spreading. At $t_0 + \tau_w$ agents with $y_M$ signals join the force of withdrawals, until eventually cumulative withdrawals reach $\kappa_L$ at $t_0 + \zeta$.

For liquid banks, things are different. No agents withdraw immediately after $t_0$, but $y_M$ agents start withdrawing at $t_0 + \tau_w$. At $t_0 + \zeta$ those early informed $y_M$ agents realize the bank is liquid and start redepositing, which makes aggregate withdrawals decrease over time. In this range, interestingly, at the same time late informed agents are still withdrawing while early informed agents who learned that the bank is liquid begin redepositing. The intriguing empirical pattern of
simultaneous withdrawing-depositing is unique to our model with rumor-based bank runs, and we wait for future empirical work to test this implication.

4.2 Bank Run Equilibrium Given $q$

4.2.1 Two-step Procedure

Define $\tau_r \equiv \zeta - \tau_w$ as the bank’s remaining survival time when the $y_M$ agents decide to stay out of the bank. The bank run equilibrium given $q$ is determined in a straight-forward two-step procedure. Define $G(\tau_r) \equiv g(\zeta - \tau_r)$, i.e., replace $\zeta - \tau$ by $\tau_r$ in (17). From the individual $y_M$ agent’s optimal withdraw condition in (18), the equilibrium redepositing time $\tau_r^*$ must satisfy

$$G(\tau_r^*) = \frac{(\lambda(1 - \gamma) - r)e^{\lambda \tau_r^*}p_0 - (1 - p_0) \left(e^{\lambda \eta} - 1\right) \frac{\pi(r - k\delta) e^{-\delta \tau_r^*} + r \left(1 - e^{\lambda \eta} (1 - p_0)\right)}{(1 - p_0) (e^{\lambda \eta} - 1) + (e^{\lambda \tau_r^*} - 1) p_0} = 0. \quad (22)$$

One important observation emerges. In (22), the equilibrium remaining survival time $\tau_r^*$ is uniquely determined, independent of the other endogenous variables $q$ or $\zeta$. This is because in the agent’s first-order condition in (17) regarding optimal withdrawal, both the hazard rate $h(\tau_w)$ in (8) and the continuation value $\tilde{V}_O(\tau_w)$ in (16) only depend on $\tau_r = \zeta - \tau_w$.

Once we pin down $\tau_r^*$ from (22), the equilibrium survival time $\zeta^*$ follows from the aggregate withdrawal condition. Combing (20) and (21), the illiquid bank fails when

$$\kappa_L = (1 - q) \left(1 - e^{-\beta \tau_r^*}\right) + q \left(1 - e^{-\beta \zeta^*}\right). \quad (23)$$

We can now solve for the equilibrium survival time $\zeta^*$ as

$$\zeta^* = -\frac{1}{\beta} \log \left[1 - \frac{\kappa_L - (1 - q) \left(1 - e^{-\beta \tau_r^*}\right)}{q}\right]. \quad (24)$$

Before we move on, we show that $\zeta^* \leq \eta$, i.e. the illiquid bank fails before the rumor stops spreading. In fact, when we wrote down the aggregate failure condition (20), we implicitly assumed at the failure time $t_0 + \zeta^*$ there are still newly informed $y_L$ agents withdrawing, which exactly requires that the illiquid bank fails before the rumor stops spreading, i.e., $\zeta^* \leq \eta$. We have the following lemma.
Lemma 1. Generically, for the bank run equilibrium to exist so that $\zeta^* < \infty$, we must have $\zeta^* \leq \eta$, and $y_L$ agents must withdraw immediately upon hearing the rumor.\footnote{Here, a statement holds “generically” if it may not hold only for some zero measure parameter set.}

We provide intuition for why $\zeta^* \leq \eta$ holds generically. The $y_M$ agent’s first-order condition in (22), pins down the equilibrium remaining survival time $\tau^*_r$ based on model primitives. However, from (21), $\tau^*_r = \zeta^* - \tau_w$ also determines the cumulative withdrawals of $y_M$ agents from the illiquid bank. Hence an extra degree of freedom is needed to ensure the consistency of both equilibrium conditions. Interestingly, the degree of freedom comes from the presence of $y_L$ agents who withdraw immediately after hearing the rumor and are still withdrawing when the bank fails. Because the required $y_M$ withdrawal is the gap between the capacity $\kappa_L$ and the cumulative $y_L$ withdrawal $q \left(1 - e^{-\beta \zeta^*}\right)$, we can find such $\zeta$ so that $y_M$ agents’ individual incentives to coincide with the required aggregate $y_M$ withdrawal. A similar argument implies that generically for the bank run equilibrium to exist, $y_L$ agents must withdraw immediately upon hearing the rumor.

4.2.2 Equilibrium Mechanism

Based on the aggregate bank failure condition, we can determine natural bounds for equilibrium remaining survival time $\tau^*_r$. When $y_M$ agents withdraw immediately after hearing the rumor $\tau_w = 0$, the remaining survival time $\zeta - \tau_w$ will assume its upper bound value:

$$\tau^*_u = \frac{1}{\beta} \ln \frac{1}{1 - \frac{\kappa_L}{q}} < \eta.$$  

On the other hand, $\zeta \leq \eta$ in Lemma 1 gives the lower bound of $\tau^*_r$:

$$\tau^*_l = \frac{1}{\beta} \ln \left(\frac{1 - q}{1 - \kappa_L - q e^{-\beta \eta}}\right).$$

Because of the parameter condition (11), $\tau^*_l > 0$ so that $y_M$ withdrawals are also contributing to the bank’s failure. We have the following key proposition.

Proposition 4. Given information acquisition effort $q$, the bank run equilibrium is characterized as follows:

1. If $G\left(\tau^*_l\right) \leq 0$, then there does not exist a bank run equilibrium.
2. If \( G(\tau_u) \geq 0 \), then the unique bank run equilibrium is \( \tau_r^* = \zeta^* = \tau_u \) and \( \tau_w^* = 0 \).

3. Otherwise, we must have \( G(\tau_u) < 0 \) and \( G(\tau_l) > 0 \), and there exists a unique bank run equilibrium \( \tau_r^* \in (\tau_l, \tau_u) \) so that \( G(\tau_r^*) = 0 \), and

\[
\zeta^* = \zeta(\tau_r^*) = \frac{1}{\beta} \ln \left[ \frac{q}{(1 - \kappa L - (1-q) e^{-\beta \tau_r^*})} \right],
\]

and \( \tau_w^* = \zeta(\tau_r^*) - \tau_r^* \). The equilibrium is stable.

The economic mechanism that pins down the equilibrium is as follows. Focus on \( y_M \) agents; and recall that \( G(\tau_r) \) in (22) gives the sign of marginal cost minus the marginal benefit of waiting a bit longer. For illustration, start with the hypothetical equilibrium where all \( y_M \) agents withdraw immediately (i.e., \( \tau_w = 0 \)); then the remaining survival time \( \tau_r = \zeta - \tau_w \) takes its upper bound \( \tau_u \).

The condition \( G(\tau_u) < 0 \) implies that this conjectured equilibrium with immediate withdrawal is not incentive compatible individually: Because the marginal benefit of waiting exceeds the marginal cost, each \( y_M \) agent would like to postpone his withdrawal.

Once \( y_M \) agents decide to wait \( \tau_w > 0 \), there are more \( y_L \) agents withdrawing before \( y_M \) agents. Then bank failure requires less cumulative mass of withdrawing \( y_M \) agents and hence a lower remaining survival time \( \tau_r = \zeta - \tau_w \) (check (23)). From (8), the more imminent failure gives rise to a higher failure hazard rate, pushing up the marginal cost of waiting. In the extreme, consider the equilibrium candidate where \( y_M \) agents wait sufficiently long, and the remaining survival time attains the lower bound \( \tau_r = \tau_l \). Then the condition \( G(\tau_l) > 0 \) implies this conjectured equilibrium is not incentive compatible either, as each individual \( y_M \) agent now wants to withdraw a bit earlier.

Combining the above two results, we can find some intermediate withdrawal time \( \tau_w^* \) with \( G(\tau_r^* = \zeta^* - \tau_w^*) = 0 \) to satisfy the individual optimality condition, and this gives the bank run equilibrium that we are after.

4.3 Endogenous Information Acquisition

Given the bank run equilibrium characterized in Proposition 4, we first characterize the individual agent’s optimal information acquisition condition. Because information quality \( q \) also affects the bank run equilibrium, this intricate feedback effect between bank runs and information acquisition may lead to multiple equilibrium.
4.3.1 The First-Order Condition of Information Acquisition

Recall that by setting $q$, we have $\Pr \{ y = y_L | t_i \} = qp(t_i | t_i)$, $\Pr \{ y = y_M | t_i \} = 1 - q$, and $\Pr \{ y = y_H | t_i \} = q (1 - p(t_i | t_i))$, while the information cost is $\chi(q) = q^2$. Denote the signal with quality $q$ as $y_q$.

When agent $t_i$ hears the rumor he spends his information collection effort to maximize the following object, taking the equilibrium survival time $\zeta^*$ as given:

$$v(0) = \Pr \{ y = y_L | t_i \} \frac{\delta}{\delta - r} + \Pr \{ y = y_M | t_i \} \frac{\delta}{\delta - r} - \alpha q^*$$

where we have used the result in Proposition 3. We are implicitly taking the bank run equilibrium here, so that the agent with $y_L$ finds immediate withdrawal optimal.

Taking the first order condition for $v(0)$ with respect to $q$, the optimal $q^*$ satisfies (recall parameter restriction (11))\(^\text{15}\)

$$\frac{p(t_i | t_i) + (1 - p(t_i | t_i))}{\mathbb{E}[V_I(0)|\text{informative signal}] - \mathbb{E}[V_I(0)|\text{uninformative signal}]} \frac{\delta}{\delta - r} - \frac{\frac{\delta}{\delta - r}}{\mathbb{E}[V_I(0)|\text{informative signal}] - \mathbb{E}[V_I(0)|\text{uninformative signal}]} - \alpha q^* \geq 0, \text{ with equality if } q^* < \frac{KL}{1 - e^{-\beta \eta}}$$

(25)

This expression is intuitive. Raising $q$ increases (lowers) the probability of (un)informative signals, but costs more. Combining with the dependence of $\zeta^*$ on $q^*$ in Proposition 4 and the individual agent’s optimal information acquisition condition (25), we can solve for the endogenous information acquisition $q^*$ and survival time $\zeta^*$ simultaneously.

4.3.2 Run and No-Run Equilibria

We show that with endogenous information acquisition, in general multiple equilibrium emerge. First, we check whether $q^* = 0$ is an equilibrium. Since the marginal cost of acquiring information is zero, in order for $q^* = 0$ to hold in equilibrium, it must be that there is no bank run.\(^\text{16}\) Of course, no bank run is also sufficient for not acquiring additional information $q^* = 0$. According to

\(^{15}\)The fact that we are focusing on bank run equilibrium and that the agent with $y_L$ finds immediate withdrawal to be optimal imply that in (25) $q^*$ cannot bind at zero, as the Blackwell information theorem implies that $p(t_i | t_i) + (1 - p(t_i | t_i)) > 0$ always, i.e., information has positive value as it improves the agent’s decision. On the other hand, the equilibrium without bank run must have $q^* = 0$ and is analyzed in Section 4.3.2.

\(^{16}\)Otherwise, with bank run immediate withdrawal with $y_L$ signals implies a positive value for the signal. See related argument in footnote 15.
Proposition 4, the condition for existence of a no-run equilibrium in this case is

$$G \left( \tau_\ell(q) \right) \big|_{q=0} \leq 0. \quad (26)$$

When (26) fails, i.e., bank run occurs even fixing \( q = 0 \) exogenously, then bank runs with positive information acquisition must exist. The next lemma summarizes this result.

**Lemma 2.** Condition (26) is a necessary and sufficient condition for the existence of an equilibrium where no bank run occurs (and therefore \( q^* = 0 \)). If (26) fails, there exist bank run equilibrium with positive information acquisition \( q^* > 0 \).

The above lemma only provides sufficient conditions for the existence of bank run equilibrium. Specifically, bank run equilibrium (with \( q^* > 0 \)) could exist even when (26) holds. Intuitively, although no-run-no-acquisition is an equilibrium, run-acquisition might also be an equilibrium. Once other agents raise the equilibrium \( q^* \) above zero, a bank run is possible, and this makes individual information acquisition self-enforcing.

In general, even among the class of bank run equilibrium with positive endogenous information acquisition, multiplicity may occur. The next lemma shows that under certain sufficient conditions provided in the Appendix, we will have at most one such equilibrium. Essentially, this condition, by guaranteeing that at the equilibrium \( q^* > 0 \) the marginal benefit of information acquisition has to go below the marginal cost for \( q \) slightly above \( q^* \), implies that the resulting equilibrium (if exist) must be unique and stable.

**Lemma 3.** Under condition (30) provided in the Appendix, the bank-run equilibrium with positive endogenous information acquisition, if one exists, is unique and stable.

The next proposition follows from the two above lemmas.

**Proposition 5.** Under condition (26) and the condition in Lemma 3, we have at most two equilibrium. The first is a no-run equilibrium without information acquisition, i.e., \( q^* = 0 \). The other is the bank run equilibrium with information acquisition so that \( q^* > 0 \) and \( \zeta^* \leq \eta \).

From now on, to facilitate analysis, we will assume the conditions of Proposition 5 hold, so we have at most two equilibrium: one equilibrium where agents do not acquire information and also
do not run, while the other equilibrium where agents acquire information aggressively and also run on the illiquid bank. In particular, we will be interested in parameters that eliminate the bank run equilibrium.

4.4 Comparative Statics

We perform comparative static analysis in this section. The following analysis focuses on the run equilibrium with endogenous information acquisition.

4.4.1 Information Acquisition Cost $\alpha$

When each agent finds it easier to acquire information, naturally the precision $q^*$ increases (the right panel in Figure 4). As shown in the left panel in Figure 4, this further leads to a shorter survival of the illiquid bank, i.e. $\zeta^*$ becomes lower. A lower $\zeta^*$ motivates each agent to learn even more aggressively, and this feedback mechanism amplifies the initial effect of a lower $\alpha$. The dashed line in Figure 4 graphs the endogenous information quality $q^*$ by fixing the survival time at its baseline level. When $\alpha$ drops by one percent from the baseline level 0.7 to 0.693, the bank failure feedback mechanism leads agents to acquire 1.2 percent more information than otherwise.

This effect suggests information acquisition complementarities among depositors. Hellwig and Veldkamp (2009) show that information acquisition exhibits complementarity if and only if the...
actions are complementary. In our model, although running on the bank is complementary, sequential information acquisitions do not necessarily exhibit complementarity. The substitutability naturally arises in the sequential learning setting through the dependence of the posterior belief on the survival time of the bank, which is to be analyzed in (27) shortly. When other depositors acquire more information, the equilibrium survival time $\zeta^*$ is shorter. Then, upon hearing the rumor, the mere survival of the bank conveys more information about the bank’s liquidity, and hence informed agents perceive the bank to be stronger. Substitutability naturally arises because this effect discourages each individual depositor’s motivation for information acquisition.

### 4.4.2 Rumor Spreading Rate $\beta$ and Awareness Window $\eta$

When the rumor spreading rate $\beta$ increases, all else equal, the illiquid bank will fail faster. This effect is illustrated in Figure 5. In response, each individual agent acquires more information, and the illiquid bank fails even faster. The feedback effects as discussed before are also present here, and this result is intuitive.

Relative to Abreu and Brunnermeier (2003), our model decouples the rumor spreading rate from the awareness window. When we turn to the effect of the awareness window $\eta$ on the equilibrium survival time $\zeta^*$ and information precision $q^*$, a surprising result emerges. When $\eta$ increases so that everybody knows that potentially there will be more informed agents attacking the bank, each individual agent acquires information less aggressively and the illiquid bank survives longer.
This surprising result not only comes from our decoupling of these two effects, but also relies on the novel uncertainty structure that we introduce in this framework. One can see the intuition by investigating the posterior probability that the bank is illiquid upon hearing the rumor and observing that the bank is still alive:

\[
p(t_i|t_i) = \frac{\Pr \{ \text{illiquid bank survives at } t_i|\kappa_L, t_i \} \Pr \{ \kappa_L \}}{\Pr \{ \kappa_L \}} = \frac{e^{\lambda \kappa_L} - 1}{e^{\lambda \eta} - 1} p_0 \frac{e^{\lambda \kappa_L} - 1}{e^{\lambda \eta} - 1} p_0 + 1 - p_0 \tag{27}
\]

When \( \eta \) is large, \( t_0 \in [t_i - \eta, t_i] \) could occur a long time ago, and the probability

\[
\Pr \{ \text{illiquid bank survives at } t_i|\kappa_L, t_i \} = \frac{e^{\lambda \kappa_L} - 1}{e^{\lambda \eta} - 1}
\]

is lower. Consequently, conditional on the bank being alive at \( t_i \), the bank is more likely to be liquid. Without uncertainty (\( p_0 = 0 \) or 1), \( p(t_i|t_i) \) does not depend on the awareness window \( \eta \).

This result differs from the casual intuition that runs are more severe with more prone-to-run agents, with the premise that bank failure requires a sufficient mass of running depositors. However, our result shows that when cumulative informed agents are enough to run down the bank, artificially shortening the awareness window will gives rise to a novel information effect with the exact opposite direction in a dynamic setting.

A Short Awareness Window Example: Subprime Mortgage Crisis On May 17, 2007 Fed Chairman Bernanke indicated in a speech about the subprime mortgage market that looser lending standards were pervasive especially in loans originated in 2006 (Bernanke, 2007). The speech took place at a time when low teaser rates on many adjustable-rate mortgages were set to expire, suggesting that the rise in defaults was just the tip of an iceberg. Subsequently, asset-backed commercial paper (ABCP) conduits experienced the modern-day equivalent of a bank “run” as ABCP outstanding dropped from $1.3 trillion in July 2007 to $833 billion in December 2007 (Acharya, Schnabl, and Suarez, forthcoming).

One interpretation within our model of this speech is that it signaled that the awareness window was relatively short, in which case there was not much to learn from the survival of ABCP conduits up to that point in time. Suppose that instead, the analysis revealed that looser lending standards were pervasive in loans originated since 2003. In that case, information about the resulting weakness...
must have been spreading for a long time. Having observed that the ABCP market kept growing despite this fact, investors would conclude that the probability of the system being liquid enough is high and a run might not occur.

5 Policy Analysis and Extensions

How much cash reserves should banks hold to prevent runs? In Section 5.1 we analyze this major question, of concern to both banks and their regulators. We then study two extensions of our base model. In Section 5.2 we introduce fundamentally insolvent banks, so that it is also socially efficient for individual agents to acquire information. Section 5.3 then considers a two-bank economy where competition amplifies the individual agent’s socially wasteful information production. For both extensions we discuss government policy during the recent crisis.

5.1 How to Eliminate Bank Runs?

In Proposition 5 we have shown that similar to the static Diamond and Dybvig setting, generally in our model there are two equilibrium: one equilibrium without learning and no bank runs, and the other with active learning and bank runs. However, the dynamic run equilibrium derived in our setting has a qualitatively different nature relative to the one in the static Diamond and Dybvig setting. This section explores this difference by focusing on the policy implications of how to eliminate the bank run equilibrium.

5.1.1 Minimum Reserve and Deposit Insurance

Unlike the typical static setting where runs occur if bank reserves are below all potential withdrawals, our rich dynamic setting with endogenous learning gives a non-trivial reserve threshold that eliminates the run equilibrium. In other words, due to uncertainty about the bank illiquidity and other depositors’ withdrawal timings, the minimum reserve requirement to fence off the bank run equilibrium in our model might be far below the level that is sufficient to cover all potential withdrawals. The same applies to deposit insurance: in our model, to eliminate runs, we only require a sufficiently high, but not full, deposit insurance.

In Figure 6, we plot the minimum reserve level $\kappa_L$ and minimum recovery rate $\gamma$ so that the
bank run equilibrium does not exist. We plot the threshold $\kappa_L$ as a function of $\gamma$. Intuitively, when depositors expect to recover more of their deposit in case of failure (larger $\gamma$) the bank is less susceptible to runs. We can interpret $\gamma$ as the level of deposit insurance. The figure reveals that in our dynamic setting, bank runs can be prevented even with partial deposit insurance ($\gamma < 1$).

In Figure 6 we also show that the minimal liquidity level to eliminate the run is much lower than the potential mass of informed agents. From the view of static Diamond and Dybvig runs, the potential mass of informed agents is the relevant liquidity reserve required to eliminate runs. However, due to the asynchronous nature of our rumor-based setting, even without information acquisition, immediate withdrawal after hearing the rumor might not be an equilibrium since the bank survives for a while. Thus, eliminating dynamic rumor-based runs requires much less liquidity reserve than the one suggested by the static perspective.

Finally, it is clear that for the model to generate quantitatively meaningful policy recommendations on banks’ capital reserve requirement, one needs reasonable parameters to begin with. As a first step toward this goal, our stylized model falls short on this dimension. However, though beyond the scope of the current paper, given the right data, a calibration (or estimation) of our model can certainly achieve this more ambitious goal.
5.1.2 Comparison to Static Bank Run Models

One potential criticism to the above discussion is that we have so far only focused on the class of equilibrium where only informed agents run on the bank. What if uninformed agents run on the bank as well? Interestingly, in our dynamic setting with uncertainty, we can rule out one class of synchronized sun-spot type bank run equilibrium (a la Diamond and Dybvig).

Proposition 6. There does not exist a synchronized pure-strategy equilibrium where all agents (including the uninformed) run at some arbitrary but fixed time $t^{SSR} \geq 0$, where “SSR” stands for sun-spot-runs.

The idea of this result is simple. If all agents coordinate on some positive time $t^{SSR} > 0$ to run on the bank, and running indeed dominates staying, then everyone wants to preempt to run earlier at $t^{SSR} - \epsilon$ to recover the full amount of their deposit. Hence under pure strategy equilibrium, the sun-spot-run time $t^{SSR}$ must be zero. But nobody will run at $t^{SSR} = 0$, because at that time it is common knowledge that the bank is liquid, thus staying is the strictly dominant strategy, and the result in Proposition 6 follows. There might exist other equilibrium where uninformed agents take mixed strategies, and it is an interesting future research topic to study the interaction between our bank run equilibrium where only informed agents run gradually and this class of sunspot equilibrium where everyone runs.

So far we have compared our model with the Diamond and Dybvig model. To our knowledge no existing studies generate a result similar to the minimum liquidity reserve in Figure 6 based on static bank run models. While it may be possible to do so using the global games technique (e.g. Goldstein and Pauzner, 2005), the mechanism will be drastically different. To illustrate this point, note that the minimum liquidity reserve can be interpreted as liquid short-term assets held by the bank in the Diamond and Dybvig framework. Clearly, the portfolio share of liquid assets affects each individual depositor’s payoff through some specific transformation technology between the liquid short-term asset and illiquid long-term asset. This effect shows up in both static and dynamic settings. However, there is one extra effect unique to the dynamic setting with learning. In the static bank run setting, withdrawals occur immediately. In contrast, in the dynamic model considered here with gradual withdrawal, a greater share of liquid holding also implies that it takes longer to run down the bank. Hence, through the endogenous learning channel, an increase in
liquidity can reduce the hazard rate of failure and cause agents to *wait* longer. If we increase liquidity just enough to make the *last* agent whose withdrawal exhausts the liquid reserve to wait long enough, the bank run is averted. This practically relevant mechanism depends on both the dynamic nature of the withdrawal timing and the endogenous learning about a possible bank failure.

### 5.2 Insolvent Banks and Stress Tests

Thus far we have focused on runs on fundamentally solvent but potentially illiquid banks, where information is socially “bad.” Of course, in practice information may be “good” because of the presence of fundamentally insolvent banks so that runs on them are socially efficient. To address this issue, we introduce insolvency into our model. We then argue that through stress tests the planner may mitigate running on fundamentally *solvent* banks by helping individual agents spot insolvent banks more easily.

#### 5.2.1 Setup

Suppose that after the liquidity event that occurs at $t_0$, the bank in our model might be insolvent, which randomly fails (rather than matures) with intensity $\xi > r$. In this event deposits recover 0 for each dollar, and therefore $\xi > r$ implies that the bank is indeed insolvent. Of course the bank can also be solvent, and if so it can be either liquid or illiquid as we modeled before.

Naturally, there will be two layers of information. The first layer of information is regarding the bank’s solvency which is both socially and individually valuable. The second is information regarding the bank’s liquidity condition, which is individually valuable (and socially destructive) when agents realize that runs on the illiquid bank become a concern. We will stress that although these two layers of information are different, they are inevitably related when individual agents are acquiring them. As shown shortly, this way we endogenize individuals endowed liquidity information precision via the active information collection about bank solvency.

Now when agent $t_i$ hears the rumor, the possibility of insolvency motivates him to spend a fixed amount of effort $e > 0$ to obtain a signal $1_z \in \{0, 1\}$; for simplicity, this signal $1_z$ perfectly reveals whether the bank is solvent or not.\(^{17}\) We focus on situations where, in equilibrium, the agent always finds it optimal to acquire this solvency signal (which is guaranteed by a sufficiently

\(^{17}\)This assumption is for clarifying the economic channel and innocuous. See detailed discussion in footnote 19.
high default intensity $\xi$). Therefore, conditional on the bank being insolvent, all agents who hear the rumor know that the bank is insolvent and therefore run on the bank immediately, and the acquisition of solvency information is socially optimal.\footnote{Suppose that the insolvent bank has a capacity of $\kappa_Z$; therefore the equilibrium failure time for insolvent bank is $t_0 + \zeta^*_Z = t_0 + \frac{1}{\beta} \ln \frac{1}{1-\kappa_Z}$.}

A by-product of the agent’s private learning about bank solvency is that he also learns something about the liquidity of the bank. We assume that given the effort $\xi$ of figuring out whether the bank is insolvent, if the bank turns out to be solvent, then the baseline quality of the bank’s liquidity signal $y$—which is the signal we modeled in Section 2.1.4—is just $\xi$. As a result, the agent’s additional liquidity information precision choice is $q \geq \xi$ with acquisition cost $\frac{a^2}{2} (q - \xi)^2$. The interpretation is that the process of collecting insolvency information inevitably teaches the agent more about the bank’s liquidity. The more the agent collects insolvency information, the more he knows about the bank’s liquidity. Our modeling that the effort of collecting insolvency information (i.e., $\xi$) becomes the baseline quality of liquidity information captures this idea in the simplest way.

5.2.2 Policy Implication: Stress Tests

The above setting has important implications for stress tests in revealing the fundamentally problematic banks. By providing insolvency information alone, the government can use stress tests to reduce $\xi$ to eliminate runs on solvent-but-illiquid banks. According to our model, if a great effort is required to learn about insolvent banks (say Lehman), i.e., a higher $\xi$, then each agent will be automatically endowed with significant information about the liquidity of solvent banks (say Citibank). As a result, runs on solvent banks may start, which pushes each agent to acquire even more liquidity information.

Stress tests provide transparency on potentially insolvent banks (Bernanke, 2009), and therefore reduce $\xi$. Government, by providing higher quality information about banks’ insolvency, can crowd out private acquisition of insolvency information. Because public information can be better targeted at insolvency alone, while the process of private acquisition of solvency information inevitably reveals liquidity information, public provision of solvency information helps all agents know that other agents do not have superior information regarding banks liquidity situation. Therefore, our model suggests that the public provision of insolvency information indirectly reduces the socially
wasteful information acquisition regarding liquidity, and therefore make runs on illiquid banks less likely.

The perfect revelation of insolvent bank clarifies the channel that we are emphasizing. Releasing better solvency information helps illiquid banks, but it is not through a higher average bank quality once the stress test isolates those insolvent ones. Rather, the channel is through the strategic interaction of individual agents as now everybody knows that everybody will wait to see the stress test and therefore not scramble to search the insolvent banks. As a result everybody will have less precise information on which solvent bank is less liquid and susceptible to a run.

This view is consistent with the Federal Reserve Board’s recent break from the traditional supervisory view of opaqueness in favor of more public disclosure to restore the confidence of investors. However, our rationale is different than the one described by Bernanke (2010) who argues that transparency allows for scrutiny by outside analysts, which enhances the credibility of the tests. Instead, we argue that by providing more information, the government crowds out the information collection effort by individuals about the solvency of the banks. Since liquidity and solvency are tightly related, this government policy has the useful by-product of reducing information collection about bank liquidity and therefore reducing the incidence of runs on illiquid banks.

Figure 7 plots the minimal illiquid bank capacity \( \kappa_L \) that eliminates the bank run equilibrium as a function of the information collection effort \( \xi \). We see from Figure 7 that as the government provides more information about the solvency of the bank (lower \( \xi \)), less liquid banks (lower \( \kappa_L \)) can avoid runs.

5.3 Multiple Solvent Banks and Policy Implications

We now investigate the model with competing solvent banks. Instead of holding cash, a bank run in this setting involves the transfer of deposited funds from one bank to another. We show that

\footnote{When the planner varies \( \xi \), agents can always perfectly spot insolvent banks, therefore the channel of insolvency information is shut down. In fact, our model can handle the case that insolvent banks are imperfectly revealed. As shown in Lemma 1, in our model generically a bank run equilibrium exists only when \( y_L \) agents withdraw immediately, and the equilibrium analysis is identical if we instead assume that agents cannot perfectly tell insolvent banks from illiquid ones (so they will withdraw immediately once receiving \( y_L \) signal or \( 1_z = 1 \)). However, under this alternative assumption, it is quite obvious that better solvency information helps illiquid banks since it allows individual agents to tell them apart from insolvent ones.}
We plot the minimal illiquid bank capacity $\kappa_L$ that eliminates the bank run equilibrium as a function of the information collection effort $\varepsilon$. Parameter values are $r = 0.09$, $\beta = 1$, $\theta = 1.03$, $\eta = 2$, $p_0 = 0.8$, $\delta = 0.12$, $k = 10^{-6}$, $\kappa_L = 0.65$, $\alpha = 0.7$, $\gamma = 0.75$.

Greater information quality leads agents to inefficient runs on otherwise solvent banks. Here, the two banks’ difference might be minuscule, and in fact transfers between the two institutions only involve social losses (transaction fees and bank failure). The planner thus can inject noise into the system, so that individual agents with less informative signals are more likely to stay in their original bank without knowing which one is the (more) liquid one.

5.3.1 Analysis

Suppose there are two banks, ex-ante identical except that half the population deposits in bank $A$ and half deposits in bank $B$. Both banks promise the same rate of return $r$. However, transferring funds between banks requires a transaction cost $k$. A liquidity event occurs at a random time $t_0$, and the rumor starts that exactly one of the banks is illiquid ($0 < \kappa_L < 1$) while the other is liquid ($\kappa_H > 1$). The prior probability that each bank is illiquid is $p_0 = 0.5$ since they are ex-ante identical. The learning process in the two bank set-up is simpler, because the passage of time without a failure teaches agents nothing about the relative viability of their bank.

As in the setup above, agents are allowed to acquire a costly signal $\tilde{y} \in \{y_L, y_M, y_H\}$ about the status of their bank with probability distribution as in (2). Agents who receive the $y_H$ signal know

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20Some anecdotal evidence that differences in depositor perception about bank liquidity motivates them to withdraw from illiquid banks is provided by Sidel, Enrich, and Fitzpatrick (2008): “Melody Williams, 50 years old, said in the past 30 days she has moved about $25,000 out of Washington Mutual, spreading it to other financial institutions she thought were stronger, including Wells Fargo & Co. Ms. Williams, the controller for an architecture firm, said she thought that Washington Mutual had gotten ‘too big for their britches’ with too many deals over the years.”
that their bank is the liquid one and therefore never withdraw. Agents who receive the $y_M$ signal gain no useful information, and due to transaction cost staying in their original bank is optimal. Finally, agents who draw the $y_L$ signal run on their bank immediately. From their perspective, the value of a dollar in their bank falls and the value of a dollar in the competing bank increases to a riskless $\frac{\delta}{\delta - r}$. Thus, as long as the transaction cost $k$ is small enough, immediate withdraw is optimal. Information is (privately) more valuable in this setup since the outside option is a nearly identical bank rather than holding cash. The following proposition characterizes the equilibrium in this setting with two banks.

**Proposition 7.** Under the two banks setup, the bank run equilibrium $\{\zeta^*, q^*\}$ is determined by the following two equations:

$$\zeta^* = -\frac{1}{\beta} \ln \left(1 - \frac{\kappa_L}{q^*}\right), \text{ and } \frac{1}{2} (1 - k) \frac{\delta}{\delta - r} + \frac{1}{2} \frac{\delta}{\delta - r} - V_I(0|y_M; \zeta^*, q^*) = \alpha q^*,$$

where the expression for $V_I(0|y_M; \zeta^*, q^*)$ is given in the Appendix.

### 5.3.2 Injecting Noise

In this extension, higher information quality $q$ about the liquidity of two solvent banks is socially undesirable, as it shortens the survival time of the illiquid bank by inducing more agents to realize that one bank strictly dominates the other. Injecting noise can alleviate the problem, and the simplest interpretation of injecting noise is to raise the information acquisition cost $\alpha$.

A bank run equilibrium requires that withdrawals by the $y_L$ agents alone can destroy a bank, i.e. $\kappa_L < q \left(1 - e^{-\beta \eta}\right)$. The threshold $q$ so that no run would occur is $q \equiv \frac{\kappa_L}{1 - e^{-\beta \eta}}$. If the planner raises $\alpha$ so that the marginal benefit of acquiring information is below its marginal cost, i.e.,

$$\frac{1}{2} (1 - k) \frac{\delta}{\delta - r} + \frac{1}{2} \frac{\delta}{\delta - r} - V_I(0|y_M) \leq \alpha q,$$

then the illiquid bank is always sufficiently liquid to sustain a run. Figure 8 plots the minimal illiquid bank capacity $\kappa_L$ that eliminates the bank run equilibrium as a function of the information

21 More specifically, we require $k$ is sufficiently small that $V_I(0|y_M) < (1 - k) \left(1 - \frac{\kappa_L}{1 - e^{-\beta \eta}}\right) < \frac{\delta}{\delta - r}$.

22 In this two-bank setting we no longer impose the restriction of $q \leq \frac{\kappa_L}{1 - e^{-\beta \eta}}$ as in condition (11), because now only $y_L$ agents are withdrawing to potentially take down the illiquid bank.
We plot the minimal illiquid bank capacity $\kappa_L$ that eliminates the bank run equilibrium as a function of the information cost $\alpha$ for the two bank setup. Parameter values are $r = 0.09$, $\beta = 1$, $\theta = 1.03$, $\eta = 2$, $\delta = 0.12$, $k = 10^{-6}$, $\kappa_L = 0.3$, $\alpha = 1$, $\gamma = 0.75$.

cost $\alpha$ for the two bank setup. The injection of noise into the economy (higher $\alpha$) blurs the differences between the competing solvent banks. The noise reduces the equilibrium information quality $q$, and as a result, a run is easier to eliminate and requires less reserves by the illiquid bank.

Consider the recent financial crisis in 2008. A fear that some banks were insolvent prompted the Capital Purchase Program commonly known as the bailout of the nine largest U.S. financial institutions on October 13, 2008. When presenting the program to the CEOs of the 9 banks, Secretary of Treasury Henry M. Paulson was concerned that the strongest banks (e.g. JPMorgan) would not participate. To make sure that they do, government officials suggested that if a bank refused the funds, its regulator would later force it to raise capital anyway and under worse terms. The government was in fact injecting noise about the liquidity of competing solvent banks into the economy. By pooling banks together the incentive to transfer funds between them was kept low enough so that none of the nine banks suffered a run.

23“I was concerned about Jamie Dimon, because JPMorgan appeared to be in the best shape of the group, and I wanted to be sure he would accept the capital.” - Paulson (2010)

24“Look, we’re making you an offer,” I said, jumping in. “If you don’t take it and sometime later your regulator tells you that you are undercapitalized and you have to raise private-sector capital but you are unable to do so, you may not like the terms if you have to come back to me.” - Paulson (2010)
6 Conclusion

We study above new dimensions of bank runs, previously unexplored, by focusing on information acquisition in rumor-based runs. These dimensions include the spreading rate of information and the awareness window over which it spreads. We show that individuals acquire information excessively about the liquidity of banks subject to runs, study the minimum liquidity reserve to eliminate bank runs which is far below that of traditional static bank run models, and analyze the role government information policy can play in bank run prevention.

Empirically, our model can shed new light on both traditional bank runs and debt runs more broadly. Our model makes new predictions linking the likelihood of a run to the cost of acquiring information, the spreading rate of the rumor and its awareness window. It also predicts that for banks that survive a run we should observe a period of simultaneous withdrawals and redeposits by heterogeneously informed agents. Our model is rich enough for structural estimation or calibration of its parameters given detailed withdrawals data, which can result in meaningful policy implications.

Finally, the dynamic bank run model we provide above can shed new light on other economic settings such as arbitrageur behavior, currency attacks, and R&D investment games.

References


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A Appendix A

A.1 Proof of Proposition 1

We have two cases to consider for the hazard rate. First, suppose \( \zeta > \eta \). Then an agent informed at \( t_i \) learns nothing from the fact that the bank has not failed by \( t_i \). His distribution of failure dates \( t_i + \tau \) is

\[
\Pi(t_i + \tau|t_i) = \begin{cases} 
0 & \tau < \zeta - \eta \\
\frac{\rho_0 e^{\lambda(\zeta - \tau)}}{e^{\lambda \eta} - 1} & \zeta - \eta \leq \tau < \zeta \\
\rho_0 & \zeta \leq \tau.
\end{cases}
\]

with non-zero density \( \pi(t_i + \tau|t_i) = \rho_0 \frac{e^{\lambda(\zeta - \tau)}}{e^{\lambda \eta} - 1} \) for \( \zeta - \eta < \tau < \zeta \). On the other hand, if \( \zeta \leq \eta \), then agent \( t_i \)'s distribution of failure dates \( t_i + \tau \) is

\[
\Pi(t_i + \tau|t_i) = \begin{cases} 
0 & \tau < 0 \\
\rho_0 & 0 \leq \tau < \zeta \\
\rho_0 \frac{1 - e^{-\lambda \tau}}{1 - e^{-\lambda \zeta}} & \zeta \leq \tau.
\end{cases}
\]

non-zero density \( \pi(t_i + \tau|t_i) = \rho_0 \frac{1 - e^{-\lambda \tau}}{1 - e^{-\lambda \zeta}} \) for \( 0 < \tau < \zeta \). Plugging either of the pairs into the definition of the hazard rate (7) yields after some algebraic manipulation the same expression (8) for any \( \tau \geq \max \{0, \zeta - \eta\} \) and zero elsewhere.

A.2 Proof of Proposition 2

We first establish the following lemma.

Lemma 4. The function \( g(\tau) \) crosses zero from below at most once in the interval \([0, \zeta]\).

Proof: Since it is the change in the numerator that dominates around \( g(\tau) = 0 \), it suffices to show that the numerator of \( g(\tau) \) (ignoring the constant)

\[
(\lambda (1 - \gamma - r) e^{\lambda (\zeta - \tau)} p_0 - (1 - p_0) \left( e^{\lambda \eta} - 1 \right) \frac{r(r - k\delta)}{\delta - r}) e^{-\delta (\zeta - \tau)}
\]  

(28)

is increasing over the interval \([0, \zeta]\). Furthermore, since \( \lambda (1 - \gamma - r) < \lambda (1 - \gamma - r (1 - k)) < 0 \) from (12), it follows that (28) is concave in \( \tau \). Let \( \tau^* \) be the unique maximizer. At the maximum

\[
\zeta - \tau^* = \frac{1}{\lambda + \delta} \ln \left( \frac{\delta (1 - p_0) \left( e^{\lambda \eta} - 1 \right) \frac{r(r - k\delta)}{\delta - r}}{\lambda (r - \lambda (1 - \gamma)) p_0} \right) < 0
\]

due to (12). Thus, the function in (28) attains its maximum to the right of \( \zeta \) and is therefore increasing over \([0, \zeta]\).
Lemma 4 implies that if $g(\zeta) \leq 0$ ($g(0) \geq 0$) then $g(\tau) < 0$ ($g(\tau) \geq 0$) always for $\tau \in [0, \zeta]$. We next consider the optimal strategy for the three cases of the proposition:

Case 1. If $g(\zeta) \leq 0$, then it is optimal to stay in the bank always. To prove our claim, it suffices to show that $V_I(\tau) > V_O(\tau)$ for $\tau \in [0, \zeta]$. Suppose not, then there must exist some $\tau_w$ so that $V_I(\tau_w) = V_O(\tau_w)$ and $V_I'(\tau_w) > V_O'(\tau_w)$ because $V_I(\zeta) > V_O(\zeta)$. From HJB equations, we have

$$h(\tau_w)(1 - \gamma) - rV_I(\tau_w) = V_I'(\tau_w) - V_O'(\tau_w) > 0$$

However, since $V_I(\tau_w) \geq V_O(\tau_w) \geq \tilde{V}_O(\tau_w)$ by definition (the first inequality is because there is no transaction cost to take one dollar out, and the second inequality is because $\tilde{V}_O(\tau_w)$ may be derived under suboptimal policy), we have

$$h(\tau_w)(1 - \gamma) - rV_I(\tau_w) < h(\tau_w)(1 - \gamma) - r\tilde{V}_O(\tau_w) = g(\tau_w) \leq 0,$$

a contradiction.

Case 2. If $g(0) \geq 0$, then it is optimal to withdraw at 0 and redeposit right after $\zeta$. Well, if $g(0) \geq 0$, then $g(\tau) \geq 0$ always for $\tau \in [0, \zeta]$, and $g(\zeta) > 0$. Using $g(\zeta) > 0$, we first show that since $k$ is arbitrarily small, there exists some $\tilde{\tau}$ close to $\zeta$ so that $V_O(\tilde{\tau}) = V_I(\tilde{\tau})$. To show this, we show that $V_O(\zeta) - V_I(\zeta)$ is strictly below zero when $k$ is arbitrarily small. To see this, from the HJB equations we know that

$$V_O(\zeta) - V_I(\zeta) = h(\zeta)(V_O(\zeta) - V_I(\zeta) + \gamma - 1) + \delta(V_I(\zeta) - V_O(\zeta)) + rV_I(\zeta)$$

$$= -g(\zeta) + (h(\zeta) + \delta - r)(V_O(\zeta) - V_I(\zeta))$$

The first is strictly negative while the second term converges to zero as $k \to 0$. Therefore, when $k$ is arbitrary small there exists some $\varepsilon$ so that $V_I(\zeta - \varepsilon) < V_O(\zeta - \varepsilon)$. Due to continuity and the fact that $V_I(\zeta) = \frac{V_O(\zeta)}{1 - k} > V_O(\zeta)$, there exists some $\tilde{\tau}$ close to $\zeta$ so that $V_I(\tilde{\tau}) = V_O(\tilde{\tau})$. Note that $V_O(\tilde{\tau}) = \tilde{V}_O(\tilde{\tau})$.

Now to prove that "it is optimal to withdraw at 0 and redeposit right after $\zeta"," we only need to show that $V_I(\tau) = V_O(\tau)$ holds for all $\tau \in [0, \tilde{\tau}]$ (intuitively, at any point of time a dollar inside the bank has the value of taking outside, it is always optimal to keep the money outside). Suppose that this does not hold; since $V_I(\tau) \geq V_O(\tau)$ in general, there must exits some point $\tau_w \in [0, \tilde{\tau}]$ so that $V_I(\tau_w) = V_O(\tau_w)$ and $V_I'(\tau_w) < V_O'(\tau_w)$. Choosing the largest value $\tau_w$, so that $V_O(\tau_w) = \tilde{V}_O(\tau_w)$ holds (i.e., the optimal continuation strategy is wait outside the bank until $\zeta$). Similar to the argument before, we have

$$h(\tau_w)(1 - \gamma) - rV_O(\tau_w) = V_I'(\tau_w) - V_O'(\tau_w) < 0,$$

but this contradicts the fact that $h(\tau_w)(1 - \gamma) - rV_O(\tau_w) = h(\tau_w)(1 - \gamma) - r\tilde{V}_O(\tau_w) \geq 0$ since $g(\tau) \geq 0$ always.

Case 3. It follows from the Lemma 4 that $g(\zeta) > 0$ and $g(0) < 0$ imply that there exists a unique $\tau_w \in (0, \zeta)$ so that $g(\tau_w) = 0$, $g(\tau) > 0$ for $\tau \in (\tau_w, \zeta)$ and $g(\tau) < 0$ for $\tau \in (0, \tau_w)$. Following the same argument in the second part by replacing 0 with $\tau_w$, we know that it is optimal to withdraw at $\tau_w$ and redeposit at $\zeta +$, and $V_I(\tau_w) = V_O(\tau_w) = \tilde{V}_O(\tau_w)$. Then to prove our claim we only need to show that $V_I(\tau) > V_O(\tau)$ for $\tau \in (0, \tau_w)$.

Let $H(\tau) \equiv V_I(\tau) - V_O(\tau)$ with $H(\tau_w) = 0$, we need to show that $H(\tau) > 0$ for $\tau \in (0, \tau_w)$, as $H(\tau) = V_I(\tau) - V_O(\tau) \geq 0$ in general. First, we show that it is impossible to have $H(\tau) = 0$ uniformly on any interval $(\tau_w - \Delta, \tau_w)$ where $\Delta > 0$; if it is true then it must be that $V_I(\tau) = V_O(\tau)$ for $\tau \in (0, \tau_w)$.
\[ V_O(\tau) = \tilde{V}_O(\tau) \] on that interval so that
\[ 0 = rV_I(\tau) + h(\tau)(\gamma - V_I(\tau)) + \delta(1 - V_I(\tau)) + V_I'(\tau) \]
\[ = r\tilde{V}_O(\tau) + h(\tau)(\gamma - \tilde{V}_O(\tau)) + \delta(1 - \tilde{V}_O(\tau)) + \tilde{V}_O'(\tau) \]
\[ = r\tilde{V}_O(\tau) - h(\tau)(1 - \gamma) = -g(\tau) > 0 \]

where the first equality is (15) and third equality is using the ODE for \( \tilde{V}_O(\tau) \) with 0 = \( h(\tau)(1 - \tilde{V}_O(\tau)) + \delta(1 - \tilde{V}_O(\tau)) + \tilde{V}_O'(\tau) \). This contradiction implies that we must have \( V_I(\tau) > V_O(\tau) \) for some \( \tau \) close to \( \tau_w \). Now suppose that there exists another point \( \tau_w^1 < \tau_w \), so that \( V_I(\tau_w) = V_O(\tau_w^1) \). Take \( \tau_w^1 \) that is close to \( \tau_w \) so that \( V_I'(\tau_w) \geq V_O'(\tau_w) \). At \( \tau_w^1 \), \( V_O(\tau_w^1) \) must satisfy the HJB in (18) (because \( \tau_w^1 \) is in the inaction region around the neighborhood, i.e., \( 1 - k \) \( V_I(\tau) < V_O(\tau) \) for \( \tau \) close to \( \tau_w \)). Then
\[ 0 = rV_I(\tau_w^1) + h(\tau_w^1)(\gamma - V_I(\tau_w^1)) + \delta(1 - V_I(\tau_w^1)) + V_I'(\tau_w^1) \]
\[ \geq rV_O(\tau_w^1) + h(\tau_w^1)(\gamma - V_O(\tau_w^1)) + \delta(1 - V_O(\tau_w^1)) + V_O'(\tau_w^1) \]
\[ = rV_O(\tau_w^1) - h(\tau_w^1)(1 - \gamma) \geq -g(\tau_w^1) > 0 \]

where we have used the HJB for \( V_O(\tau_w^1) \), and the fact that \( V_O(\tau_w) \geq \tilde{V}_O(\tau_w^1) \) in general. Again we get a contradiction with (15).

### A.3 Proof of Proposition 3

Given \( \tau_w \) simple integration yields
\[ V_I(0|y_M) = \left[ \frac{\delta e^{\lambda(1-p_0)} - 1}{\delta-r} + e^{-\delta-\tau_w} \left( e^{\lambda \eta} - 1 - e^{\lambda \eta} e^{\lambda(\zeta-\tau_w)} \right) p_0 \right] V_O(\tau_w) \]
\[ e^{\lambda \eta} - 1 - e^{\lambda \eta} p_0 + e^{\lambda \eta} p_0 \]

Note that \( h(\tau_w)(1 - \gamma) = rV_O(\tau_w) \) implies that
\[ \left( e^{\lambda \eta} - 1 - e^{\lambda \eta} e^{\lambda(\zeta-\tau_w)} \right) p_0 V_O(\tau_w) = \frac{\lambda(1 - \gamma) e^{\lambda(\zeta-\tau_w)} p_0}{r} \]

where we used the definition of \( h \) in (7). Then note that
\[ \frac{\lambda(1 - \gamma)}{r} - \frac{\delta + \lambda \gamma}{\lambda + \delta - r} = \frac{(\lambda + \delta)(\lambda(1 - \gamma) - r)}{r(\lambda + \delta - r)} \]

which gives our expression.

### A.4 Proof of Lemma 1

Suppose that \( \zeta > \eta \), so that at \( \zeta \) the cumulative withdrawal from \( y_L \) agents is \( q \left( 1 - e^{-\beta \eta} \right) \).
Then using the aggregate condition (23), we can back out the equilibrium \( \tau_r \) for \( y_M \) agents as

\[
\tau_r = -\frac{1}{\beta} \ln \left( 1 - \frac{\kappa_L - q \left( 1 - e^{-\beta \eta} \right)}{1 - q} \right).
\]

However, unless parameters are such that the above \( \tau_r \) happens to satisfy \( G(\tau_r) = 0 \) which is the \( y_M \) agents’ optimal waiting decision, generically this cannot occur.

Now we rule out the case that \( y_L \) agents wait a positive time \( \tau_w^L > 0 \). Clearly, \( y_M \) agents will set \( \tau_w^M \) \( > \tau_w^L \). Importantly, their respective redepositing times \( (\tau_w^M \) and \( \tau_r^L \)) have to satisfy the individual optimality conditions in the nature of (22), which determine \( \tau_w^M \) and \( \tau_r^L \) fully. However, generically these \( \tau_w^M \) and \( \tau_r^L \) will not satisfy the aggregate withdrawal condition: either both agents are withdrawing at \( \zeta \) so that the aggregate withdrawal condition is \( \kappa_L = (1 - q) \left( 1 - e^{-\beta \tau_r^M} \right) + q \left( 1 - e^{-\beta \tau_r^L} \right) \); or at \( \zeta \) there are no \( y_L \) agents withdrawing then the aggregate withdrawal condition is \( \kappa_L = (1 - q) \left( 1 - e^{-\beta \tau_r^M} \right) + q \left( 1 - e^{-\beta \eta} \right) \).

### A.5 Proof of Proposition 4

First note that the \( G \) function is the mirror image of individual FOC condition function, i.e., \( G(\tau_r) = g(\zeta - \tau_r) \), and it shares the same (but opposite) property of \( g(\cdot) \) shown in Lemma 4:

**Corollary 1.** \( G(\tau_r) \) crosses zero from above at most once on \( \tau_r \in [0, \zeta] \).

This result implies that the following holds for the three cases of the proposition:

*Case 1.* If \( G(\tau_r^L) \leq 0 \), then \( G(\tau_r) \leq 0 \) for all \( \tau_r \geq \tau_r^L \). Thus if all other agents strategy is to redeposit after any \( \tau_r \geq \tau_r^L \), it is optimal for the individual agent to deviate and wait a bit longer. Therefore, \( \zeta^* \to \infty \) and no run equilibrium exists.

*Case 2.* If \( G(\tau_r^u) \geq 0 \), then \( G(\tau_r) \geq 0 \) for all \( \tau_r \leq \tau_r^u \). Thus, if all other agents’ strategy is to withdraw at some interior \( \tau_r \leq \tau_r^u \), it is optimal for the individual agent to deviate and withdraw earlier. Therefore, agents withdraw immediately in the only symmetric equilibrium.

*Case 3.* Finally, if \( G(\tau_r^u) < 0 \) and \( G(\tau_r^L) > 0 \) then by continuity of \( G \) and Corollary 1, there exists a unique bank run equilibrium \( \tau_r^* \in (\tau_r^L, \tau_r^u) \) so that \( G(\tau_r^*) = 0 \). Plugging into (23) we get the equilibrium survival time \( \zeta^* \) and waiting time \( \tau_w^* \). A second implication of Corollary 1 is that \( G'(\tau_r^*) < 0 \). Therefore the equilibrium is stable.

### A.6 Proof of Lemma 2

The proof of the first statement is given in the text. To show the second statement, we need to show that the agent’s FOC in information acquisition

\[
A(q) = p(t_i|t_i) + (1 - p(t_i|t_i)) \frac{\delta}{\delta - r} - V_I(0|y_M) - \alpha q \geq 0 \text{ with equality if } q < \frac{\kappa_L}{1 - e^{-\beta \eta}},
\]

combined with bank-run equilibrium condition, has a solution. Note that both \( p(t_i|t_i) \) and \( V_I(0|y_M) \) (given in (27) and Proposition 3) depend on \( \zeta \) that is determined in Proposition 4 about bank run equilibrium given \( q \). Now because (26) fails, the bank run equilibrium exists even with exogenously
given \( q = 0 \). From (19) it is optimal to withdraw given \( y_L \). The agent’s strategy depends on his
signal, and therefore information has a positive value, i.e. \( A(0) = p(t_i|t_i) + (1 - p(t_i|t_i)) \frac{\delta}{\delta - r} - V_l(0|y_M) > 0 \). Now suppose that \( q \) takes its upper bound \( \frac{\kappa L}{e^{\delta q} - \kappa L} \); if \( A\left( q = \frac{\kappa L}{e^{\delta q} - \kappa L} \right) > 0 \) then
the upper bound information quality and associated bank run equilibrium is just the equilibrium
that we are after. If instead \( A\left( q = \frac{\kappa L}{e^{\delta q} - \kappa L} \right) < 0 \), then an interior equilibrium exists because of the
continuity of \( A(q) \).

### A.7 Proof of Lemma 3

First, when the bank run equilibrium occurs with corner solution \( \zeta = \tau^*_r = \frac{1}{\beta} \ln \frac{1}{1 - \kappa L} \), then the
marginal benefit of information \( MB = p(t_i|t_i) + (1 - p(t_i|t_i)) \frac{\delta}{\delta - r} - V_l(0|y_M) \) is independent of \( q \),
and the equilibrium \( q^* \) equates \( MC = \alpha q^* = MB \). Therefore the equilibrium is unique and stable
(\( MB \) is constant while \( MC \) increases with \( q \)). Also if \( q^* \) takes the upper bound corner value, the
associated run equilibrium is unique as well. So the rest of proof focus on the case where both the
information quality of equilibrium survival time take interior solutions.

From now on we focus on interior bank run equilibrium. Importantly, this implies that \( \tau_r \)
is determined in (22) which only depends on primitives. Therefore we treat \( \tau_r \) as a primitive
parameter. The FOC (25) when the agent sets \( q^* \) is

\[
\begin{align*}
&\left[ (e^{\lambda \zeta} - 1) p_0 + (1 - p_0) (e^{\lambda \eta} - 1) \frac{\delta}{\delta - r} - \delta e^{\eta (1 - p_0) - 1} \right] \\
&\quad \times (1 - e^{-(\delta - r) \tau_w}) e^{\lambda \tau_r} p_0 \left( \frac{(\lambda + \delta)(1 - \gamma) - r}{r(\lambda + \delta - r)} \right) - \alpha q (1 - p_0) (e^{\lambda \eta} - 1) + (e^{\lambda \zeta} - 1) p_0 \\
&= 0
\end{align*}
\]

where \( \zeta \) and \( \tau_w = \zeta - \tau_r \) depend on \( q \) through (24). We have

\[
\zeta' \left( q \right) = \tau'_w \left( q \right) \frac{e^{-\beta \zeta} - e^{-\beta \tau_r}}{q \beta e^{-\beta \zeta}} = \frac{1 - e^{\beta \tau_w}}{q \beta} < 0.
\]

The derivative at the point where (29) takes zero value yields:

\[
\begin{align*}
&\lambda \zeta e^{\lambda \zeta} p_0 - \delta (e^{\lambda \eta} (1 - p_0) - 1) e^{-(\delta - r) \tau_w} \left[ \frac{\lambda (1 - \gamma) - r}{\lambda + \delta - r} - \alpha q e^{\lambda \zeta} \zeta' \left( q \right) \right] e^{\lambda \tau_r} p_0 \left( \frac{(\lambda + \delta)(1 - \gamma) - r}{r(\lambda + \delta - r)} \right) \\
&\quad + \alpha (1 - p_0) (e^{\lambda \eta} - 1) + (e^{\lambda \zeta} - 1) p_0 \\
&= \lambda \zeta e^{\lambda \zeta} p_0 \left[ \frac{\lambda (1 - \gamma) - r}{\lambda + \delta - r} - \alpha q \right] + e^{-(\delta - r) \zeta} \zeta' \left( q \right) e^{\lambda \tau_r} p_0 \left( \frac{(\lambda + \delta)(1 - \gamma) - r}{r(\lambda + \delta - r)} \right) - \delta (e^{\lambda \eta} (1 - p_0) - 1) \\
&\quad - \alpha (1 - p_0) (e^{\lambda \eta} - 1) + (e^{\lambda \zeta} - 1) p_0 \\
&= \zeta' \left( q \right) e^{-(\delta - r) \zeta} \left[ \alpha (\lambda + \delta - r) \zeta' \left( q \right) \left( \frac{\lambda (1 - \gamma) - r}{\lambda + \delta - r} - \alpha q \right) \right] + e^{\lambda \tau_r} p_0 \left( \frac{(\lambda + \delta)(1 - \gamma) - r}{r(\lambda + \delta - r)} \right) - \delta (e^{\lambda \eta} (1 - p_0) - 1) \\
&\quad - \alpha (1 - p_0) (e^{\lambda \eta} - 1) + (e^{\lambda \zeta} - 1) p_0 \\
\end{align*}
\]

The second line is clearly negative. Assume that (note that \( e^{\lambda \eta} (1 - p_0) < 1 \)

\[
\begin{align*}
e^{(\delta - r) \tau_r} &\left[ e^{\lambda \tau_r} p_0 (\delta - r) \left( \frac{(\lambda + \delta)(1 - \gamma) - r}{r(\lambda + \delta - r)} \right) - \delta (e^{\lambda \eta} (1 - p_0) - 1) \right] \\
&+ \lambda e^{(\lambda + \delta - r) \eta} p_0 \left[ \frac{\lambda (1 - \gamma) - r}{\lambda + \delta - r} - \alpha \frac{\kappa L}{1 - e^{\delta \eta}} \right] > 0.
\end{align*}
\]

Then, since \( \zeta < \eta \), \( q^* < \frac{\kappa L}{1 - e^{\delta q} - \kappa L} \), and \( \zeta' \left( q \right) < 0 \), the first line is also negative. As a result,
the derivative of (29) is always negative. As a result, when (29) equals zero, it must go down.
Combined with differentiability of (29), this result rules out multiple solutions, because if there exist, then there must have one solution with the local slope being nonnegative. Therefore, (29) crosses zero at most once from above, which implies that the bank run equilibrium, if exists, is unique and stable.

A.8 Proof of Proposition 5

Now suppose that $q^* = \frac{\kappa L}{1-e^{-\beta \eta}}$ which is the upper bound, so that $\tau_r = 0$. Suppose that

$$\left(\lambda (1 - \gamma) - r\right) p_0 - (1 - p_0) \left( e^{\lambda \eta} - 1 \right) \frac{r (r - k \delta)}{\delta - r} + r \left( 1 - e^{\lambda \eta} (1 - p_0) \right) \geq 0$$

so that run equilibrium occurs. There must be another smallest $q$ so that run could occur. The lower bound $q$ satisfies $G \left( \tau_r \left( q \right) \right) = 0$, i.e., $q$ solves

$$(\lambda (1 - \gamma) - r) \left( \frac{1 - q}{1 - \kappa L - q e^{-\beta \eta}} \right)^{\frac{1}{\beta}} p_0 - (1 - p_0) \left( e^{\lambda \eta} - 1 \right) \frac{r (r - k \delta)}{\delta - r} \left( \frac{1 - q}{1 - \kappa L - q e^{-\beta \eta}} \right)^{-\frac{1}{\beta}} + r \left( 1 - e^{\lambda \eta} (1 - p_0) \right) = 0.$$ 

At this $q$, $\zeta = \eta$ and $\tau_w = \eta - \frac{1}{\beta} \ln \left( \frac{1 - q}{1 - \kappa L - q e^{-\beta \eta}} \right)$, and we need to check if agents have sufficient incentives to improve signal quality (note that in this case $p \left( t_i \mid t_i \right) = p_0$ because $\zeta = \eta$)

$$p_0 + (1 - p_0) \frac{\delta}{\delta - r} - V_1 \left( 0 \mid y_M \right) > \alpha q^*$$

If this is true, then we will have another point $q^+ > q$ to be the bank run equilibrium. Otherwise, each individual agent would like to lower their information $q$ below $q$—but we know lower $q$ cannot trigger bank run, contradiction. Moreover, it implies that at $q$, $\bar{F}OC = MB - MC < 0$. Given the following conditions, FOC has to goes down once it touches zero, so there will be no $q > q$ to generate bank run equilibrium. The FOC in (25) is

$$\frac{\left( e^{\lambda \zeta} - 1 \right) p_0 + (1 - p_0) \left( e^{\lambda \eta} - 1 \right) \frac{\delta}{\delta - r} - \delta \left( e^{\lambda \eta (1 - p_0)} - 1 \right) \left( 1 - e^{\delta - r}) \tau_w \right) \left( \lambda + \delta \eta - \delta (\lambda (1 - \gamma) - r) \right) \left( 1 - e^{\delta - r}) \tau_w \right)}{\left( 1 - p_0 \right) \left( e^{\lambda \eta} - 1 \right) + \left( e^{\lambda \zeta} - 1 \right) p_0} - \frac{\alpha q^* \left( 1 - p_0 \right) \left( e^{\lambda \eta} - 1 \right) + \left( e^{\lambda \zeta} - 1 \right) p_0}{\left( 1 - p_0 \right) \left( e^{\lambda \eta} - 1 \right) + \left( e^{\lambda \zeta} - 1 \right) p_0}$$

we know that that $\tau_r$ is independent of $q$, and

$$\zeta \in \left[ \frac{1}{\beta} \ln \frac{1}{1 - \kappa L}, \eta \right]$$

and

$$\tau_w \in \left[ 0, \eta - \tau_r \right]$$

and

$$\zeta' \left( q \right) = \tau_w \left( q \right) = e^{-\beta \tau_w} \frac{1 - e^{\beta \tau_w}}{q \beta e^{-\beta \zeta}} = \frac{1 - e^{\beta \tau_w}}{q \beta} \frac{1}{q \beta} = \frac{1 - e^{\beta \eta - \tau_r}}{q \beta}$$

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Now we put bounds on each term. we have

\[ \lambda c' e^{\lambda c} p_0 - \delta \left( e^{\lambda \eta} (1 - p_0) - 1 \right) e^{-(\delta - r)\tau w} \tau_w' - \frac{\delta + \lambda \gamma}{\lambda + \delta - r} p_0 \lambda e^{\lambda c} + (\delta - r) e^{-(\delta - r)w} e^{\lambda \tau r} p_0 \frac{\left( \lambda + \delta \right) \left( \lambda (1 - \gamma) - r \right)}{r (\lambda + \delta - r)} \tau_w' \]

\[ - \alpha \left( (1 - p_0) e^{\lambda \eta} - 1 \right) e^{\lambda \tau d} p_0 \left( \delta - r \right) - \delta \left( e^{\lambda \eta} (1 - p_0) - 1 \right) > 0 \]

Now we put bounds on each term. we have

\[ 0 > \lambda c' e^{\lambda c} p_0 \left[ \frac{\lambda (1 - \gamma) - r}{\lambda + \delta - r} - \alpha q^* \right] > \lambda e^{\lambda \eta} p_0 \left[ \frac{\lambda (1 - \gamma) - r}{\lambda + \delta - r} - \alpha \right] \]

and

\[ e^{-(\delta - r)\tau w} \left[ e^{\lambda \tau d} p_0 \left( \delta - r \right) \frac{\left( \lambda + \delta \right) \left( \lambda (1 - \gamma) - r \right)}{r (\lambda + \delta - r)} - \delta \left( e^{\lambda \eta} (1 - p_0) - 1 \right) \right] > e^{-(\delta - r)(\eta - \tau d)} \left[ e^{\lambda \tau d} p_0 \left( \delta - r \right) \frac{\left( \lambda + \delta \right) \left( \lambda (1 - \gamma) - r \right)}{r (\lambda + \delta - r)} - \delta \left( e^{\lambda \eta} (1 - p_0) - 1 \right) \right] \]

and finally the last term is

\[ - \alpha \left( (1 - p_0) e^{\lambda \eta} - 1 \right) + \left( e^{\lambda c} - 1 \right) p_0 < - \alpha \left( (1 - p_0) e^{\lambda \eta} - 1 \right) + \left( e^{\frac{\lambda c}{2} \ln \frac{1}{1 - \alpha}} - 1 \right) p_0 \]

therefore the FOC are bounded above by

\[ \frac{1 - e^{\lambda \eta} p_0 \left[ \frac{\lambda (1 - \gamma) - r}{\lambda + \delta - r} - \alpha \right] + e^{-(\delta - r)(\eta - \tau d)} \left[ \left( \delta - r \right) \frac{\lambda (1 - \gamma)}{r} e^{\lambda \tau d} p_0 - \delta \left( e^{\lambda \eta} (1 - p_0) - 1 \right) - \frac{\delta + \lambda \gamma}{\lambda + \delta - r} p_0 \left( \delta - r \right) e^{\lambda \tau d} \right] \}}{\lambda \eta} \]

\[ - \alpha \left( (1 - p_0) e^{\lambda \eta} - 1 \right) + \left( e^{\frac{\lambda \eta}{2} \ln \frac{1}{1 - \alpha \eta}} - 1 \right) p_0 \]

if this quantity is negative, then there is only one interior solution.

A.9 Proof of Proposition 6

First, suppose to the contrary that everyone runs at some \( t_{SSR} > 0 \), which must be because the bank has some strictly positive probability of being illiquid at \( t_{SSR} \) (otherwise waiting is optimal). But if the bank is indeed potentially illiquid at \( t_{SSR} \), then every agent would like to preempt. It is because conditional on the bank being illiquid, running at \( t_{SSR} \) (with others) gives the agent \( \kappa L + \gamma (1 - \kappa L) < 1 \), while he will receive his entire deposit if he preempts the run and withdraws at \( t_{SSR} - \epsilon \) for some \( \epsilon > 0 \). This argument implies that the only possible synchronized run occurs at \( t_{SSR} = 0 \). However, at \( t_{SSR} = 0 \), it is common knowledge that the bank is liquid almost surely, as the bank starts liquid and only becomes illiquid at some liquidity event \( t_0 > 0 \) (recall Section
2.1.3), i.e. with probability 1, at \( t^{SSR} = 0 \) we have and \( \kappa = \kappa_H > 1 \). Thus, even if everyone else
withdraws immediately, an individual agent would remain in the bank and this behavior is not an
equilibrium.

A.10 Proof of Proposition 7

At the time of hearing the rumor, the value of a dollar in the bank for agents with \( y_L \) and \( y_H \)
signals are respectively \( V_I (0|y_L) = \frac{(1-k)\delta}{\delta - r} \) and \( V_I (0|y_H) = \frac{\delta}{\delta - r} \). In order to calculate the value for
agents with the \( y_M \) signal, note that with probability 1/2 the original bank is the illiquid one, but
the agent can deposit his funds (after the liquidation cost \( 1 - \gamma \)) to the liquid one. As a result, the
value with \( y_M \) signal is

\[
V_I (0|y_M) = \frac{1}{2} \int_{\{0\}}^{\frac{\kappa}{\lambda}} \left[ \delta e^{-(\delta - r)s} (1 - \Pi(s)\kappa_L) + e^{-(\delta - r)s} \pi(s)\kappa_L \gamma (1 - k) \frac{\delta}{\delta - r} \right] ds + \frac{\delta}{2 \delta - r}
\]

\[
= \frac{1}{2} \int_{0}^{\frac{\kappa}{\lambda}} \left[ \delta e^{-(\delta - r)s} \left( 1 - \frac{1 - e^{-\lambda s}}{1 - e^{-\lambda L}} + e^{-(\delta - r)s} \frac{\lambda e^{-\lambda s}}{1 - e^{-\lambda L}} \gamma (1 - k) \frac{\delta}{\delta - r} \right) ds + \frac{\delta}{2 \delta - r}
\]

\[
= \frac{1}{2} \int_{0}^{\frac{\kappa}{\lambda}} \left[ -e^{-\lambda s} \frac{\delta}{\delta - r} (1 - e^{-(\delta - r)q}) + \frac{\delta}{\delta - r + \lambda} \left( 1 - e^{-(\delta - r)q} \right) \gamma (1 - k) \frac{\delta}{\delta - r} \right] ds + \frac{\delta}{2 \delta - r}
\]

\[= \frac{1}{2} \int_{0}^{\frac{\kappa}{\lambda}} \left[ -e^{-\lambda s} \frac{\delta}{\delta - r} (1 - e^{-(\delta - r)q}) + \frac{\delta}{\delta - r + \lambda} \left( 1 - e^{-(\delta - r)q} \right) \gamma (1 - k) \frac{\delta}{\delta - r} \right] ds + \frac{\delta}{2 \delta - r}
\]

\[= \frac{1}{2} \int_{0}^{\frac{\kappa}{\lambda}} \left[ -e^{-\lambda s} \frac{\delta}{\delta - r} (1 - e^{-(\delta - r)q}) + \frac{\delta}{\delta - r + \lambda} \left( 1 - e^{-(\delta - r)q} \right) \gamma (1 - k) \frac{\delta}{\delta - r} \right] ds + \frac{\delta}{2 \delta - r}
\]

B Appendix B

We consider the non-stationary part of the model here. If \( t_0 < \eta \), then some early informed
agents with \( t_i < \eta \) knows that \( t_0 \in [0, t_i] \), and this truncation implies strategy may be \( t_i \)-dependent.
However, as shown in Abreu and Brunnermeier (2003), those early agents will be bunching together
to eliminate the non-stationarity. We modify their results to our setting.

Focus on the bank being illiquid. To be precise, follow Abreu and Brunnermeier (2003) in our
model the agents who hears the rumor before \( \zeta - \tau_w = \tau_r \) will behave as if the agent who hears the
rumor exactly at \( \tau_r \). The strategy of the agent who hears rumor at \( \tau_r \) is that, independent of signal
\( y_L \) or \( y_M \) he will withdraw at \( \tau_r + \tau_w = \zeta \). Moreover, for agents who hear rumor at \( t_i \in [\tau_r, \zeta] \),
take the following strategy. If they receive \( y_L \) signal then he will withdraw at \( \zeta \), while if they
get \( y_M \) signal then they withdraw at \( t_i + \tau_w \). This additional modification is because relative to
Abreu and Brunnermeier (2003) agents may have different signals in our model.

For illustration, suppose that \( t_0 = 0 \) so the bank should fail at \( \zeta \). Recall that there are \( q \)
measure of \( y_L \) signals and \( 1 - q \) measure of \( y_M \) signals. Since the information keeps spreading at \( \zeta \) (recall that
\( \eta < \zeta \)) and all agents hears the rumor before \( \zeta \) will withdraw at \( \zeta \), there are \( q \left( 1 - e^{-\beta \zeta} \right) \) measure
of \( y_L \) agents withdrawing. On the other hand, \( y_M \) agents who hear the rumor in the interval \([0, \tau_r] \)
are withdrawing at \( \zeta \), with a total mass of \( (1 - q) \left( 1 - e^{-\beta \tau_r} \right) \). Therefore, we have

\[
(1 - q) \left( 1 - e^{-\beta \tau_r} \right) + q \left( 1 - e^{-\beta \zeta} \right) = \kappa_L,
\]

which is exactly (23). A similar argument can be applied to the case of \( t_0 > 0 \) so that the bank
failure time is \( t_0 + \zeta \): this is because endogenously there are less agents bunching at the physical
time \( \zeta \), so the bank failure time is postponed to \( t_0 + \zeta > \zeta \).

There is one issue that our richer (than Abreu and Brunnermeier (2003)) setting leads to poten-
tial non-stationarity. Although withdraw behavior can be stationary, the endogenous learning about bank liquidity is non-stationary when $\eta < \zeta$. In fact, initially when $t_0 = 0$, agents have no other information so $p(t_i|t_i) = p_0$ must holds. In stationary state, $p(t_i|t_i) = \tilde{p}_0 \equiv \frac{(e^{\lambda \zeta} - 1)p_0}{(1 - p_0)(e^{\lambda \eta} - 1) + (e^{\lambda \zeta} - 1)p_0} < p_0$. This difference potentially alters the optimal withdraw strategies for agents with different timings. To resolve this issue, we simply assume that for $t_i < \eta$, the prior is time-varying

$$p_0(t_i) = \frac{(e^{\lambda t_i} - 1)p_0}{[(1 - p_0) (e^{\lambda \eta} - 1) + p_0 (e^{\lambda t_i} - 1)]},$$

and one can show that with this specification, the resulting posterior upon hearing the rumor, $p(t_i|t_i)$, is always $\tilde{p}_0$. One can presumably achieve this by a more structural way; for instance, introduce other shocks so that, conditional on survival and hearing the rumor, the posterior of the bank being illiquid is always $\tilde{p}_0$. Also, we have to fix the signal quality structure the $q$ in (2) is the same as in the stationary phase. We deem these technical issue non-essential for the economic questions that we are after in this paper.